Oral Presentation

DIFFERENT UNDERSTANDINGS OF MATHEMATICS:

AN EPISTEMOLOGICAL APPROACH TO BRIDGE THE GAP BETWEEN SCHOOL AND UNIVERSITY MATHEMATICS

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A survey in Germany amongst students who have chosen to obtain a teaching degree shows that the transition from school to university mathematics is experienced in the context of a major revolution regarding their views about the nature of mathematics. Motivated by the survey, a team of researchers is currently working on a historically-motivated concept for an undergraduate course to help bridge the gap.

THE PROBLEM OF TRANSITION: STILL OF IMPORTANCE

A classical problem of mathematics education certainly is the problem of transition from school mathematics to university mathematics and back again. It is a problem all high school teachers around the world encounter during their training. Even Felix Klein (1849-1925), prominent mathematician and mathematics educator, in this context complained about the phenomena he coined as “double discontinuity”:

> The young university student found himself, at the outset, confronted with problems, which did not suggest, in any particular, the things with which he had been concerned at school. Naturally he forgot these things quickly and thoroughly. When, after finishing his course of study, he became a teacher, he suddenly found himself expected to teach the traditional elementary mathematics in the old pedantic way; and, since he was scarcely able, unaided, to discern any connection between this task and his university mathematics, he soon fell in with the time honoured way of teaching, and his university studies remained only a more or less pleasant memory which had no influence upon his teaching. (Klein, 1908/1932, p. 1, author’s translation)

In the following we focus on the “first discontinuity”, postulating an epistemological gap between school and university mathematics. As the problem is at least more than 100 years old, definitive solutions do not seem to appear on the horizon (cf. Gueudet, 2008). Contrarily, dropout rates (especially in western countries) remain on a constantly high level – in Germany about 50% of the students studying mathematics or mathematics-related fields stop their efforts before having finished a bachelor’s degree (Heublein et al., 2012). This again leads to an at least perceived intensification of research in this field. In 2011 the most important professional associations regarding mathematics (education) in Germany (DMV-Mathematics, GDM – Mathematics Education & MNU – STEM Education) formed a task force regarding the problem of transition (cf. http://www.mathematik-schule-hochschule.de). In February 2013 a scientific conference with the topic “Mathematik im Übergang Schule/Hochschule und im ersten Studienjahr" ("Mathematics at the Crossover School/University in the First Academic Year") in Paderborn (Germany) attracted almost 300 participants giving over 80 talks regarding the problematic transition-process from school to university mathematics. The proceedings of this conference (Hoppenbrock et al., 2013) and its predecessor on special transition-courses (Bausch et al., 2014) give an impressive overview on the necessity and variety of approaches regarding this matter. Interestingly a vast majority of the studies and best practice examples for "transition-courses" locate the problem in the context of deficits (going back as far as junior high school) regarding the content knowledge of freshmen at universities.

> In the "precourse and transition course community" it seems to be consensus by now that existing deficits in central fields of lower-secondary school's mathematics make it difficult for Freshmen to acquire concepts of advanced elementary mathematics and to apply these. Fractional arithmetic, manipulation of terms or concepts of variables have an important role e.g. regarding differential and integral calculus or non-trivial application contexts and constitute insuperable obstacles if not proficiently available. (Bieler et al., 2014, p. 2, author's translation)

The question of how to provide first semester university students with the obviously lacking content knowledge is certainly an important facet of the transition problem. But as the results of an empirical study suggest, there are other, deeper problem dimensions which aid in further understanding the issue.

MOTIVATION: A SURVEY

To investigate new perspectives on the transition problem, approximately 250 pre-service secondary school teachers from the University of Siegen and the University of Cologne in 2013 were asked for retrospective views on their way from school to university mathematics. When the questionnaire was disseminated the students had been at the universities for about one year. Surprisingly, the systematic qualitative content analysis of the data (Mayring 2002; Huberman & Miles 1994) showed that from the students' point of view it was not the deficits in (the level and amount of) content knowledge that dominated their description of their own way from school to university mathematics. To a substantial extent, students reported problems with a feeling of "differentness" of school and university mathematics than with the abrupt rise in content-specific requirements. Three exemplar answers to the question,

> What is the biggest difference or similarity between school and university mathematics? What prevails? Explain your answer.

illustrate this point quite clearly.

> Student (male, 20 years): "The biggest difference is, that university mathematics is a closed logical system, constituted by proofs. School mathematics in contrast is limited to applications. Regarding the topics there are more similarities, regarding the process of reasoning more differences." (author's translation)

> Student (male, 19 years): "The fundamental difference develops as mathematics in school is taught ostensibly ("*anschaulich*"), whereas at university it is a rigid modern-axiomatic structure characterizing mathematics. In general there are more differences than similarities, caused by differing aims." (author's translation)

At this point we can only speculate on the term "aims" but in reference to other formulations in his survey it seems possible that he distinguishes between general education (*Allgemeinbildung*) as an aim for school and specialized scientific teacher-training at universities.

The third example is impressive in the same sense:

> Student (female, 20 years):

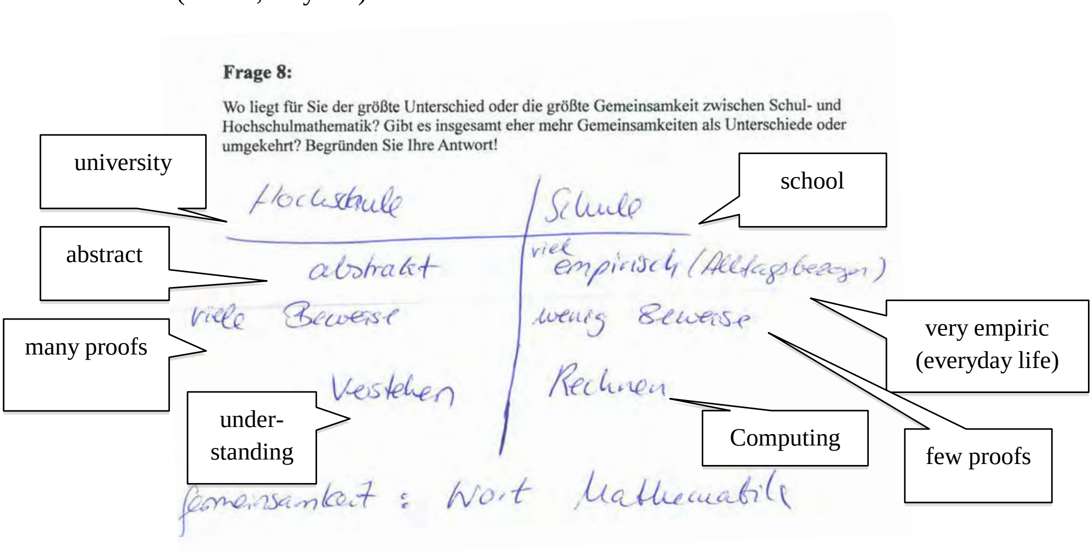


Figure 1: A student's appreciation of difference or similarity between school and university mathematics.

In all three cases the students clearly distinguish between school and university mathematics, which is most prominent in the last example (see Fig. 1): for this student school mathematics and university mathematics are so different, that the only remaining similarity is the word 'mathematics'. This "differentness" encountered by the students is specified in further parts of the questionnaire with terms as *vividness, references to everyday life, applicability to the real world, ways of argumentation, mathematical rigor, axiomatic design, etc.*

Using additional results of studies with a similar interest (e.g. Gruenwald et al., 2004; Hoyles et al., 2001) the author comes to the preliminary conclusion that pre-service teachers clearly distinguish between school and university mathematics regarding the *nature of mathematics*. In the terms of Hefendehl-Hebeker et al., the students encounter an "Abstraction shock." (Hefendehl-Hebeker et al., 2010)

This sets the framework for further research concerning the problem of transition: following the idea of constructivism in mathematics education, students construct their own picture of mathematics with the material, problems and stimulations teachers provide in the classroom or lecture hall (Anderson et al., 2000; Bauersfeld, 1992). Thus it is helpful to reconstruct the *nature of mathematics* communicated explicitly and implicitly in high school and university textbooks, lecture notes, standards, etc., with a special focus on differences.

REFLECTIONS ABOUT THE NATURE OF MATHEMATICS & MATHEMATICAL BELIEF SYSTEMS IN SCHOOL AND UNIVERSITY

Beliefs

The terms *nature of mathematics* and *belief system* regarding mathematics are closely linked to each other if we understand learning in a constructive way. Schoenfeld (1985) successfully showed that personal belief systems matter when learning and teaching mathematics:

> One's beliefs about mathematics [...] determine how one chooses to approach a problem, which techniques will be used or avoided, how long and how hard one will work on it, and so on. The belief system establishes the context within which we operate […] (Schoenfeld, 1985, p. 45)

From an educational point of view beliefs about mathematics are decisive for our mathematical behavior. For example, there are four prominent categories of beliefs concerning mathematics as a discipline distinguished by Grigutsch, Raatz, and Törner (1998): *the toolbox aspect, the system aspect, the process aspect and the utility aspect.* Liljedahl et al. (2007) specified this wide range of possible aspects of a mathematical worldview as follows:

> In the "toolbox aspect", mathematics is seen as a set of rules, formulae, skills and procedures, while mathematical activity means calculating as well as using rules, procedures and formulae. In the "system aspect", mathematics is characterized by logic, rigorous proofs, exact definitions and a precise mathematical language, and doing mathematics consists of accurate proofs as well as of the use of a precise and rigorous language. In the "process aspect", mathematics is considered as a constructive process where relations between different notions and sentences play an important role. Here the mathematical activity involves creative steps, such as generating rules and formulae, thereby inventing or re-inventing the mathematics. Besides these standard perspectives on mathematical beliefs, a further important component is the usefulness, or utility, of mathematics. (p. 279)

Very often these beliefs are located within certain fields of tension (*Spannungsfelder*): there is, for example, the *process aspect* which is always implicitly connected to its opposite pole the *product aspect*. Another pair of concepts in this sense is certainly an *intuitive aspect* on the one hand and a *formal aspect* on the other, having even a historical dimension: "There is a problem that goes through the history of calculus: the tension between the intuitive and the formal." (Moreno-Armella, 2014, p. 621) These fields of tension may help to describe the problems the students encounter on their way to university mathematics. Especially helpful when looking at the survey results, representing one important facet, seems to be the tension between what Schoenfeld calls an *empirical belief system* and a *formal(istic) belief system* – a convincing analytical distinction following the works of Burscheid and Struve (2010). The *empirical belief system* on the one hand describes a set of beliefs in which mathematics is understood as an experimental natural science, which of course includes deductive reasoning, about empirical objects. Good examples for such a belief system can be found in the history of mathematics. The famous mathematician Moritz Pasch (1843-1930) who completed Euclid's axiomatic system, explicitly understood geometry in this way,

> The geometrical concepts constitute a subgroup within those concepts describing the real world […] whereas we see geometry as nothing more than a part of the natural sciences. (Pasch, 1882, p. 3)

Mathematics in this sense is understood as an empirical, natural science. This implies the importance of inductive elements as well as a notion of truth bonded to the correct explanation of physical reality. In Pasch's example Euclidean geometry is understood as a science describing our physical space by starting with evident axioms. Geometry then follows a deductive buildup – but it is legitimized by the power to describe the physical space around us correctly. This understanding of mathematics as an empirical science (on an epistemological level) can be found throughout the history of mathematics – prominent examples for this understanding are found in many scientists of the 17^{th} and 18^{th} centuries. For example, Leibniz conducted analysis on an empirical level; the objects of his *calculus differentialis* and *calculus integralis* were curves given by construction on a piece of paper – not as today's abstract functions (cf. Witzke, 2009).

Now, how does all of this come together with students and the transition problem? If we take a closer look at the survey results, and combine this with a look at current textbooks we see that students at school are likely to acquire an empirical belief system – which on epistemological grounds shows parallels to the historical understanding of mathematics. These epistemological parallels were fundamental for the design of our 'transition seminar' for students. The main idea is that the recognition and appreciation of different conceptions of mathematics in history (i.e., those held by expert mathematicians) can help students to become aware of the own belief system and may guide them to make necessary changes.

SCHOOL & UNIVERSITY

If we look at the most recent National Council of Teachers of Mathematics (NCTM) standards and prominent school books we see that for good reasons, mathematics is taught in the context of concrete (physical) objects at school: The process standards of the NCTM and in particular "connections" and "representations" (which are comparable to similar mathematics standards in Germany) focus on empirical aspects. At school it is important that students “recognize and apply mathematics in contexts outside of mathematics” or “use representations to model and interpret physical, social, and mathematical things” (NCTM, 2000, p 67).

The prominent place of illustrative material and visual representations in the mathematics classroom has important consequences for the students’ views about the *nature of mathematics*. Schoenfeld proposed that students acquire an *empiricist belief system* of mathematics at school (Schoenfeld, 1985; 2011). This is caused by the fact that mathematics in modern classrooms does not describe abstract entities but a universe of discourse ontologically bounded to “real objects”: *Probability Theory* is bounded to random experiments from everyday life, *Fractional Arithmetic* to “pizza models”, *Geometry* to straightedge and compass constructions, *Analytical Geometry* to vectors as arrows, *Calculus* to functions as curves (graphs) etc.).

At university things can look totally different. Authors of prominent textbooks (in Germany as well as in the U.S.) for beginners at university level depict mathematics in quite a formal rigorous way. For example in the preface of Abbott’s popular book for undergraduate students, *Understanding Analysis*, it becomes very clear where mathematicians see a major difference between school and university mathematics: “Having seen mainly graphical, numerical, or intuitive arguments, students need to learn what constitutes a rigorous proof and how to write one” (Abbott, 2000, p. vi). This view is also transported by Heuser’s popular analysis textbook for first semester students (Heuser, 2009, p. 12, author’s translation).

> The beginner at first feels […] uncomfortable […] with what constitutes mathematics:
>
> - The brightness and rigidity in concept formation
> - The pedantic accurateness when working with definitions
> - The rigor of proofs which are to be conducted […] only with the means of logic not with *Anschauung*.
> - Finally the abstract nature of mathematical objects, which we cannot see, hear, taste or smell. […]

This does not mean that there are no pictures or physical applications in the book; it is common sense that modern mathematicians work with pictures, figural mental representations and models – but in contrast, to many students it is clear to them that these are illustrations or visualizations only, displaying certain logical aspects of mathematical objects (and their relations to others) but by no means representing the mathematical objects in total. This distinction gets a little more explicit if we look at a textbook example. In school books the reference objects for functions are curves. Functions are virtually identified with empirically given curves. Consequently, schoolbook authors work with the concept of graphical derivatives in the context of analysis (see Fig. 2). At university, curves are by no means the reference objects anymore; they are only one possibility to interpret the abstract notion of function. The graph of a function in formal university mathematics is only a set of (ordered) pairs.

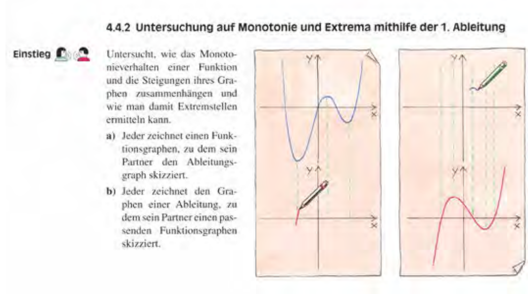


Figure 2: Graphical derivatives in a schoolbook.

If we, in a theoretical simplification, contrast the *empirical belief system* many students acquire in classroom with the *formal(ist) belief system* students are supposed to learn at university we have one model that explains the problem of transition. For example, in this model the notion of proof differs substantially in school and university mathematics. Whereas at universities (especially in pure mathematics) only formal deductive reasoning is acceptable, non-rigorous proofs relying on "graphical, numerical and intuitive arguments" are an essential part of proofs in school mathematics where we explain phenomena of the "real world". In the terminology of Sierpinska (1992), students at this point have to overcome a variety of "epistemological obstacles", requiring a big change in their understanding of what mathematics is about.

HELPING TO BRIDGE THE GAP: SEMINAR CONCEPTION

The findings of the survey and the theoretical discussion are essential for the author's design of a course for pre-service teachers which will be taught, evaluated and analyzed for the first time in spring 2015 together with the University of Cologne (Horst Struve) and the Florida State University (Kathleen Clark). [1]

The overall aim of the course is to make students aware and to lead them to understand of crucial changes regarding the nature of mathematics from school to university. The different conceptions of mathematics in school and university can be reconstructed as well for the history of mathematics, as we previously stated. Thus, an understanding of how and why changes regarding the nature of mathematics (for example from empirical-physical to formal-abstract) took place may be achieved by an historical-philosophical analysis (cf. Davies 2010). This is the key idea of the course. Thereby we hope that the students then can link their own learning biographies to the historical development of mathematics. This conceptual design of the course draws upon positive experience with explicit approaches regarding changes in the *belief system* of students from science education (esp. "Nature of Science", cf. Abd-El-Khalick & Lederman, 2001).

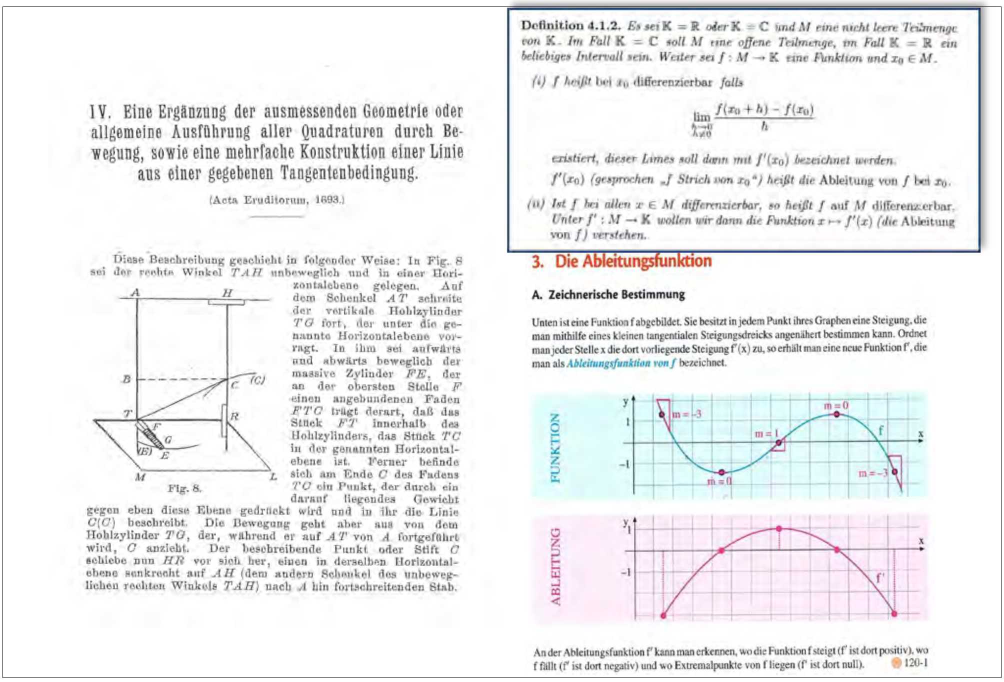
The undergraduate course designed to cope with the transition problem is organized in four parts:

1) Raising attention for the importance of beliefs/philosophies of mathematics.
2) Historical case study: geometry from Euclid to Hilbert. Which questions lead to the modern understanding of mathematics?

3) What characterizes modern formal mathematics? (Exploration of Hilbert's approach.)
4) Summarizing discussion and reflection

1) *Raising attention for the importance of beliefs/philosophies of mathematics.*

In the first part of the seminar we want to make the students aware of the idea of different belief systems/natures of mathematics. Here we start with individual reflections and work with authentic material such as transcripts from Schoenfeld's research that clearly show the meaning and relevance of the concept of an empirical belief system. In a next activity we will compare textbooks: University textbooks, school textbooks, and historical textbooks.



IV. Eine Ergänzung der ausmessenden Geometrie oder allgemeine Ausführung aller Quadraturen durch Bewegung, sowie eine mehrfache Konstruktion einer Linie aus einer gegebenen Tangentenbedingung.

(Acta Eruditorum, 1693.)

Diese Beschreibung geschieht in folgender Weise: In Fig. 8 sei der rechte Winkel TAH unbeweglich und in einer Horizontalebene gelegen. Auf dem Schenkel AT schreite der vertikale Hohlzylinder TG fort, der unter die genannte Horizontalebene vorragt. In ihm sei aufwärts und abwärts beweglich der massive Zylinder FE, der an der obersten Stelle F einen angebundenen Faden FTC trägt derart, daß das Stück FT innerhalb des Hohlzylinders, das Stück TC in der genannten Horizontalebene ist. Ferner befinde sich am Ende C des Fadens TC ein Punkt, der durch ein darauf liegendes Gewicht gegen eben diese Ebene gedrückt wird und in ihr die Linie $C(C)$ beschreibt. Die Bewegung geht aber aus von dem Hohlzylinder TG, der, während er auf AT von A fortgeführt wird, C anzieht. Der beschreibende Punkt oder Stift C schiebe nun HR vor sich her, einen in derselben Horizontalebene senkrecht auf AH (dem andern Schenkel des unbeweglichen rechten Winkels TAH) nach A hin fortschreitenden Stab.

Definition 4.1.2. *Es sei $\mathbb{K} = \mathbb{R}$ oder $\mathbb{K} = \mathbb{C}$ und M eine nicht leere Teilmenge von $\mathbb{K}$. Im Fall $\mathbb{K} = \mathbb{C}$ soll M eine offene Teilmenge, im Fall $\mathbb{K} = \mathbb{R}$ ein beliebiges Intervall sein. Weiter sei $f : M \to \mathbb{K}$ eine Funktion und $x_0 \in M$.*

(i) f heißt bei x_0 differenzierbar *falls*

$$\lim_{\substack{h \to 0 \\ h \neq 0}} \frac{f(x_0 + h) - f(x_0)}{h}$$

existiert, dieser Limes soll dann mit $f'(x_0)$ bezeichnet werden. $f'(x_0)$ (gesprochen „f Strich von x_0") heißt die Ableitung *von f bei x_0.*

(ii) Ist f bei allen $x \in M$ differenzierbar, so heißt f auf M differenzierbar. *Unter $f' : M \to \mathbb{K}$ wollen wir dann die Funktion $x \mapsto f'(x)$ (die* Ableitung *von f) verstehen.*

3. Die Ableitungsfunktion

A. Zeichnerische Bestimmung

Unten ist eine Funktion f abgebildet. Sie besitzt in jedem Punkt ihres Graphen eine Steigung, die man mithilfe eines kleinen tangentialen Steigungsdreicks angenähert bestimmen kann. Ordnet man jeder Stelle x die dort vorliegende Steigung f'(x) zu, so erhält man eine neue Funktion f', die man als *Ableitungsfunktion von f* bezeichnet.

An der Ableitungsfunktion f' kann man erkennen, wo die Funktion f steigt (f' ist dort positiv), wo f fällt (f' ist dort negativ) und wo Extremalpunkte von f liegen (f' ist dort null). 120-1

Figure 3: Three excerpts of different textbooks for comparison.

The three excerpts (Fig. 3) illustrate how we will work in this comparative setting. In the upper right-hand corner of Fig. 3 is a formal university textbook definition of differentiation. It is characterized by a high degree of abstraction: the objects of interest are functions defined on real numbers and even complex numbers. We see a highly symbolic definition where the theoretical concept of limit is necessary. Just below we see in contrast, is an excerpt from a popular German school textbook. Here the derivative function is defined on a purely empirical level: the upper curve is virtually identified with the term 'function'. Characteristic points are determined by an act of measuring and the slopes of the triangles are then plotted underneath and constitute the red curve. Interestingly for students, should be that the theoretical abstract notion of function – as it is presupposed in the university textbook – did not always characterize analysis.

If we look back to Leibniz (one of the fathers of analysis), with his *calculus differentialis* and *intergalis*, he conducted mathematics in a rather empirical way as well (cf. Witzke, 2009): his objects were curves given by construction on a piece of paper – properties like differentiability or continuity were read out of the curve…and not only there seem to be parallels on an epistemological level between school analysis and historical analysis. For example, Leibniz presented (published in 1693) the invention of the so-called integrator (left-hand side of Fig. 3), a machine that was designed to draw an anti-derivative curve by retracing a given curve. So here, as in the schoolbook, it is on an epistemological level that the empirical objects form the theory. Combined with selected quotes from schoolbooks emphasizing its experimental and empirical access to mathematics, quotes like Pasch's regarding Euclidean geometry as an empirical science on the one hand and Hilbert's statement,

> If I subsume under my points arbitrary systems of things, e.g. the system: love, law, chimney sweep ..., and then just assume all my axioms as relationships among these things, then my theorems, e.g. also the Pythagorean theorem, are true of these things, too. (Hilbert to Frege, 1980, p.13, author's translation)

on the other, it becomes clear that something revolutionary had changed regarding the nature of mathematics at the end of 19th century mathematics. This change is a revolution, which on an epistemological level has parallels to what students encounter when being faced with abstract university mathematics.

2) *Historical case study: geometry from Euclid to Hilbert. Which questions lead to the modern understanding of mathematics?*

An adequate description of the development of the conception of mathematics in the course of history requires more than one book. We refer to the following ones: Bonola (1955) for a detailed historical presentation; Grabe (2001), Greenberg (2004) and Trudeau (1995) for a lengthy historical and philosophical discussion; Ewald (1971), Hartshorne (2000), and Struve & Struve (2004) for a modern mathematical presentation. Additionally, Davis & Hersh (1981 & 1995) or Ostermann & Wanner (2012) presented aspects of the historical and philosophical discussion in a concise manner, relatively easily accessible to students.

The overall aim of the historical case study is to make students aware of how the nature of mathematics changed over history. Regarding our theoretical framework, we aim to make explicit how geometry – which for hundreds of years seemed to be the prototype of empirical mathematics, describing physical space – did develop into the prototype of a formalistic mathematics as formulated in Hilbert's foundations of Geometry in 1899 (cf. Fig.4). And thus, we can help students on their way to develop an understanding for different mathematical conceptions, in particular, modern ones taught at the university level.

In the course we start with Euclid's *Elements*: they show what a deductively built piece of mathematics looks like in a prototype manner. Here we will induce the students, e.g., to display in a graphical manner how Pythagoras' theorem can be traced down to the five postulates – as the 2013 survey results showed that most students were not familiar with a deductive structure after one year at university.

It is quite important for the overall goal of the seminar that the *Elements* give reason to discuss status, meaning and heritage of axiomatic systems. Thereby we will focus on the self-evident character of the axioms (or, postulates) describing *physical space* in a true manner – as undoubtedly provides insights on the surrounding real space which were accepted without proof (cf. Garbe, 2001, p. 77).

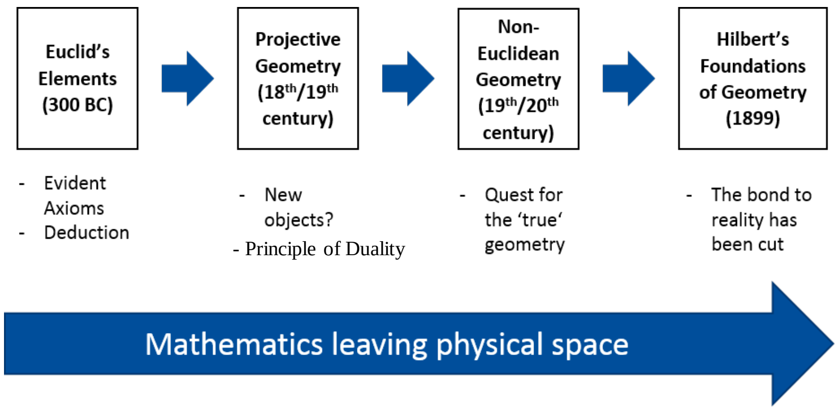


Figure 4: The historical and philosophical development of mathematics along the development of geometry

Projective geometry is the next stop on our way to a modern understanding of geometry (cf. Ostermann & Wanner, 2012, pp. 319-344). Starting with the question of whether other geometries, besides the Euclidean one, are conceivable, projective geometry seems to be an ideal case. Related to the overall aim of the course, the notion that there is more than one geometry can foster the idea that there is more than 'one' mathematics, leading away from the quest for one unique mathematics describing physical space (cf. Davis & Hersh, 1985, pp. 322-330).

Well, on the one hand, projective geometry seems to be so intuitive and evident if we look at its origin in the arts in the vanishing point perspective. On the other hand it adds new abstract objects to the Euclidean geometry (the infinitely distant points on the horizon) and familiarizes us with the idea that all parallels may meet eventually. With projective geometry the students encounter a further axiomatizable geometry – which has irritating properties that finally influenced Hilbert (cf. Blumenthal, 1935, p.

402) to ultimately design a geometry *free of any physical references*. Julius Plücker saw in the 19th century for the first time, that theorems in projective geometry hold if the terms “straight line” and “point” are interchanged: the so-called principle of duality – giving a clear hint why it became reasonable in mathematics to focus on mere structures of theories.

A decisive revolutionary step towards a formalistic abstract formulation of geometry can then be seen in the development of the so-called non-Euclidean geometries. This development is connected in particular to the names Janos Bolyai (1802-1860), Nikolai Ivanovitch Lobatchevski, Carl Friedrich Gauß (1777-1855) oder Bernhard Riemann (1826-1866) (cf. Garbe, 2001, Greenberg, 2004, Trudeau, 1995 on their historical role regarding non-Euclidean Geometries).

In fact, the non-Euclidean geometries developed from the “theoretical question” around Euclid’s fifth postulate, the so-called parallel postulate:

> Let the following be postulated: [...]
>
> That if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the straight lines, if produced indefinitely, will meet on that side on which the angles are less than two right angles. (Heath, 1908)

Compared to the other postulates like the first, “to draw a straight line from any point to any point,” the fifth postulate sounds more complicated and less evident. This postulate cannot be “verified” by drawings on a sheet of paper as parallelity is a property presupposing infinitely long lines. In the words of Davis & Hersh (1995), “it seems to transcend the direct physical experience” (p. 242). In history this was seen as a blemish in Euclid’s theory and various attempts have been undertaken to overcome this flaw. On the one hand, different individuals tried to find equivalent formulations, which are more evident (e.g. Proclus (412-485), John Playfair 1748-1819)[1]. On the other hand, several mathematicians tried to deduce the fifth postulate from the other postulates so that the disputable statement becomes a theorem (which does not need to be evident) and not a postulate (e.g. Girolamo Saccheri (1667-1733), Johann Heinrich Lambert (1728-1777)). (cf. Davis & Hersh, 1985, pp. 217-223; Garbe, 2001, pp. 51-74; Greenberg, 2004, pp. 209-238)

In contrast in the 18th and 19th century, Bolyai, Lobatchevski, Gauß, and Riemann experimented with negations and replacements of the fifth postulate guided by the question of whether the parallel postulate was logically dependent of the others (cf. Greenberg, 2004, pp. 239-248). If this would have been true – Euclidean geometry

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[1] To Proclus, who was amongst the first commentators of Euclid’ Elements in ancient Greece, already formulated doubts on the parallel postulate and formulated around 450 an equivalent formulation (cf. Wußing & Arnold 1978, p. 30). Playfair’s formulation (1795), “in a plane, given a line and a point not on it, at most one line parallel to the given line can be drawn through the point“, is quite popular today (cf. Prenowitz & Jordan 1989, p. 25; Gray 1989, p. 34).

should actually work without it – what it does, in a sense that no inconsistencies occur. But this logical act leads to conclusions that differ from those in Euclidean geometry.

For example:

- In the so-called hyperbolic geometry the sum of interior angles in a triangle adds up to less than 180°, in elliptic geometry to more than 180° (see Fig. 6).
- The ratio of circumference and diameter of a circle in hyperbolic geometry is bigger than π, in elliptic smaller than π.
- In hyperbolic as in elliptic geometry triangles which are just similar but not congruent do not exist.
- In hyperbolic geometry there is more than one parallel line through a point P to a given line g and in elliptic geometry there are no parallel lines at all (see Fig. 5).

(cf. Davis & Hersh, 1985, p. 222; Garbe, 2001, p. 59)

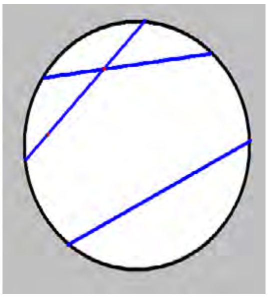

Figure 5: Klein's Model for hyperbolic geometry: More than one parallel line to a straight line through a given point.

Working with texts and sources regarding the process of discovery of the non-Euclidean geometries may have an important impact on students' beliefs system, as it tackles the so-called "Euclidean Myth" (Davis & Hersh, 1985) which was widespread within the 2013 survey results: to many first-year students mathematics is a monolithic block of eternal truth; a theorem, once proven, necessarily holds in every context.

With the discovery of the non-Euclidean geometries, it became apparent in history that there is no such truth in a total sense. In contrast, there seems to be more of such truths, depending on the context you are working in. A discussion of Gauss's qualms to publish results on non-Euclidean geometry implicitly emphasizing this aspect, afraid of being accused of doing something suspect, or the (probably legendary) story that he tried to measure on a large scale whether the world is Euclidean (cf. Garbe, 2001, pp. 81-85), can make the students amenable to the revolutionary character of this discovery for changing natures of mathematics. Following Freudenthal's (1991) idea of guided reinvention, recapitulating the history of humankind seems to bear quite fruitful perspectives for the development of individual belief systems in this context.

Finally, from the discussion of the non-Euclidean geometries students will approach the questions which lead to Hilbert's formal(istic) turn. *If there was more than one consistent geometry, which one is the true one?* This question is closely linked to the question, *what is mathematics?*

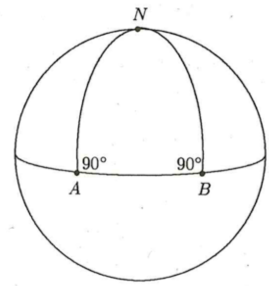

Figure 6: Angle sum in an elliptic triangle.

3) What characterizes modern formal mathematics? (Exploration of Hilbert's approach.)

Hilbert actually gave an answer to this problem – not only in a philosophical and programmatic way but also by formulating a geometry "exempla trahunt" (Freudenthal, 1961 p. 24), a discipline that was seen for ages as the natural description of physical space, in a formalistic sense and characterized by an axiomatic structure. The established axioms are fully detached and independent from the empirical world, which leads to an absolute notion of truth: mathematical certainty in the sense of consistency. With Hilbert the bond of geometry to reality is cut. This becomes very vivid when reading Hilbert's *Foundations of Geometry* (1902; see Fig. 7) in detail, as we plan to do with the students in the seminar.

§ 1. THE ELEMENTS OF GEOMETRY AND THE FIVE GROUPS OF AXIOMS.

Let us consider three distinct systems of things. The things composing the first system, we will call *points* and designate them by the letters A, B, C,. . . ; those of the second, we will call *straight lines* and designate them by the letters a, b, c,. . . ; and those of the third system, we will call *planes* and designate them by the Greek letters α, β, γ,. . . The points are called the *elements of linear geometry*; the points and straight lines, the *elements of plane geometry*; and the points, lines, and planes, the *elements of the geometry of space* or the *elements of space*.

Figure 7: The famous first paragraph of Hilbert's *Foundations of Geometry.*

Hilbert does not give his concepts an explicit semantic meaning; he speaks independently from any empirical meaning of distinct systems of things. Consequently, intuitive relations like *in between* or *congruent* do not have an empirical meaning but are relations fulfilling certain formal properties only. (cf. for example, Greenberg, 2004, pp. 103-129)

As we all know, the development of mathematics did not come to an end with Hilbert; the seminar is intended to finish with discussions of texts taken from *What is Mathematics, Really?* (Hersh, 1997). Hersh understands "mathematics as a human activity, a social phenomenon, part of human culture, historically evolved, and intelligible only in a social context" (p. xi), which creates a balanced view.

However, nobody will deny that formalism in Hilbert's open-minded version had a lasting effect on the development of mathematics. As a consequence, today's university mathematics has the freedom to be developed without being ‘true’ in an absolute sense anymore (cf. Freudenthal, 1961), but nevertheless including the possibility to interpret it physically again.

> In the meantime, while the creative power of pure reason is at work, the outer world again comes into play, forces upon us new questions from actual experience, opens up new branches of mathematics, and while we seek to conquer these new fields of knowledge for the realm of pure thought, we often find the answers to old unsolved problems and thus at the same time advance most successfully the old theories. And it seems to me that the numerous and surprising analogies and that apparently prearranged harmony which the mathematician so often perceives in the questions, methods and ideas of the various branches of his science, have their origin in this ever-recurring interplay between thought and experience. (Hilbert, 1900)

It is the openness and freedom of questions of absolute truth, which Hilbert replaced by the concept of logical consistency that made mathematics so successful in the 20th century (cf. Freudenthal, 1961, p. 24; Garbe, 2001, pp. 100-109, Tapp, 2013 p. 142). In Einstein’s words:

> Geometry thus completed is evidently a natural science; we may in fact regard it as the most ancient branch of physics. Its affirmations rest essentially on induction from experience, but not on logical inferences only. We will call this completed geometry “practical geometry,” and shall distinguish it in what follows from “purely axiomatic geometry.”[…]As far as the propositions of [modern axiomatic] mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality.[…] The progress achieved by axiomatics consists in its having neatly separated the logical-formal from its objective or intuitive content […] These axioms are free creations of the human mind. The axioms define the objects of which geometry treats. […] I attach special importance to the view of geometry, which I have just set forth, because without it I should have been unable to formulate the theory of relativity. (Einstein, 1921, as cited in Freudenthal, 1961, p. 16; for a readable article on exactly this point compare with Hempel (1945))

This makes again quite clear that modern mathematics after Hilbert is on epistemological grounds completely different than (historical) empirical mathematics and of course mathematics taught in school. Whether the first is grounded on set axioms and the notion of mathematical certainty (inconsistency), the second and third are grounded in evident axioms – thus describing physical space including a notion of (empirical) truth, resting essentially on induction from experience.

4) Summarizing discussion and reflection

In the last part of the course we want to initiate discussions connecting the insights gained from the historical perspectives with the individual biographies. We plan to remind the students about the preliminary discussions regarding different personal belief systems that occurred in the first part of the course. The intention is that the transparency on the historical problems that led to a modern abstract understanding of mathematics leads to an understanding of what happens if students live on epistemological grounds through this revolution as individuals, thus opening differentiated views on the transition problem. For school purposes – from a well-informed mathematics educator's point of view – nothing speaks against doing mathematics in an empirical way (when including deductive reasoning, of course, otherwise it would just be phenomenology). History has shown that empirical mathematics was a decent way to develop mathematical knowledge and the experimental natural sciences generate knowledge comparably. Yet approaches to bring formal(istic) mathematics into school classrooms have failed miserably. Moreover, we cannot step away from teaching mathematics in a theoretical way at universities. In contrast, the course described here intends to make tangible, understandable, and explicit (that) to first-year students the transition from school mathematics to university mathematics is an epistemological obstacle. Hefendehl-Hebeker (2013, p. 80) sees quite comparably

> […] a principle difference between school and university is at university with the axiomatic method a new level of theory formation has to be reached, and thus it follows that the discontinuity cannot be avoided.

So if the discontinuity cannot be avoided, what may teachers and students at university take from a course like the one described here?

1) The historical excursions do not only focus on the beliefs aspect but also demonstrate crucial mathematical activities – especially regarding deductive reasoning within the frameworks of consistent mathematical theories.

2) Teachers and students should be sensible about the dimension of the problem: it is not as easy as repeating some lower secondary school mathematics, as many approaches seem to suggest. Instead a revolutionary act of conceptual change is required that does not occur overnight and needs guidance. The historical questions that lead to the modern understanding of mathematics are too sophisticated and waiting for students to develop these for themselves is a particular burden on top of all the other factors of beginning at university. The approach of initiating these questions explicitly within the described framework may support a more adequate and prompt change of belief system.

3) The course should sensitize for crucial communication problems. Teachers and students should acknowledge that when talking about mathematics, using the same terms might not imply talking about the same things. For example, students may come from school to university having learned calculus in an empirical context such that functions might be equivalent to curves. This might imply that properties like continuity or differentiability are empirical and can be read from the sketched graph of the function (comparable to 17^{th} century mathematicians). The lecturer at university on the other hand probably has a general abstract notion of function implying a completely different notion of mathematical reasoning and truth. In particular, lecturers should repeatedly check if the knowledge of their students is still bonded to

(single) objects of reference. The same holds for the students eventually leaving university and starting as secondary school mathematics teachers: they should be aware that what they consider from an abstract point of view their students may instead possess visualizations of abstract notions as the reference objects.

CONCLUDING REMARKS/PERSPECTIVES

An in-depth study based on data collected from surveys containing both standardized and open-ended items and student interviews accompanying the seminar course describes here will follow in 2015. A follow-up course will be conducted at Florida State University in spring 2016. The data, along with the personal evaluations of the involved researchers, will clarify whether explicitly discussing historical epistemological obstacles regarding changes on mathematical belief systems supports students on their way through university mathematics. Much will depend on if we succeed in initiating thinking-processes which bring the historical and personal perspectives together. Only then will it be possible to determine if the historical-philosophical elements of the course have a lasting effect.

NOTES

1. Many elements of the course discussed here have been tested in isolated settings in Cologne and Siegen but not in a coherent course to face the problem of transition.

2. Also, there is another dimension to the axioms as fundamentals of a platonic construct of ideas, called “geometry”.

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