Workshop

TEACHING THE MATHEMATICAL SCIENCES IN FRANCE AND GERMANY DURING THE 18TH CENTURY: THE CASE STUDY OF NEGATIVE NUMBERS

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In the following, we provide the analysis of a particular case study in the mathematical teaching of the 18th century: how negative numbers were introduced to students, justified, and used in practice. We focus on a small selection of French and German textbooks, paying particular attention to their didactic approaches. Our main aim is to point out the similarities and differences between these presentations.

INTRODUCTION

During the 18th century, a question of concern in mathematical teaching was how negative numbers had to be interpreted, either within arithmetic or algebra (for further reading, see Schubring 2005). In the past, there had already been attempts for interpretations. Common explanations were possessions and debts, and quantities moving along in opposed directions. There are three German sources that give us an insight into the contemporary discussion and the associated problems with negative numbers. To these belong first *Gedanken über den gegenwärtigen Zustand der Mathematik* (1789) by Johann Andreas Christian Michelsen (1749-1797), second *Versuch das Studium der Mathematik durch Erläuterung einiger Grundbegriffe und durch zweckmäßigere Methoden zu erleichtern* (1805, published anonymously) by Franz Spaun (1753-1826), third the reaction on Spaun's writing, namely *Ueber Newtons, Eulers, Kästners und Konsorten Pfuschereien in der Mathematik* (1807) by Karl Christian von Langsdorf (1757-1834). Spaun criticized among other things that the meaning of the plus and minus operators have a double meaning; first representing the arithmetic operations of addition and subtraction, second as algebraic symbols for positive and negative numbers (cf. Spaun 1805, p. 7 and p. 18). This aspect concerning the opposed numbers caused difficulties for contemporaries. Spaun also spoke against the usage of the expression "negative" in order to denote negative numbers (cf. Spaun 1805, p. 7). In contrast, Langsdorf argued that this expression was a convention for mathematicians and could be used for negative numbers (cf. Langsdorf 1807, p. 12).

We consider a small selection of French and German textbooks from this period. In order to identify the selection, the following criteria had been taken into account. Firstly, we limited to textbooks written in a national language, namely French or German, during the 18th century. Secondly, we searched for textbooks that were written with a teaching purpose for higher education. Thirdly, we considered only the ones that were meant to provide a complete presentation of the mathematical sciences.[1] Afterwards, among this first raw selection, we chose some of the most renowned and used textbook, according to primary and secondary sources. In particular, for the French part, we deal with Bélidor's (1725), La Caille's (1741), Camus' (1749), Bézout's (1764, 1770), and Bossut's (1771). For the German part, we analyse Wolff's (1775) and Kästner's works. We also choose the textbook by Euler for the German case, but for the ease of the workshop in an English edition (1822). The first and original edition was written in German and published in 1770.

With our case study, we wanted to take a look at different approaches to negative numbers, and especially at their justifications, in the small selection of textbooks. After a brief presentation of the German and French circumstances (educational system, institutional conditions, position of mathematics, textbooks and their authors, …), we invited the participants to work in teams on the different sources. At the end of the workshop, every team presented their results. The aim was to show the differences among the various approaches to negative numbers at that time, also in comparison with the developments that lead to nowadays approaches. In order to make the study on the sources easier for the participants and to guarantee comparable results, we proposed the following questions for the analysis of the sources:

- Definition: Is there a definition of negative numbers? If yes, where is it in the textbook? Are there examples to explain the definition? If yes, what are they?
- Terminology: Which expressions are used?
- Are there interpretative models for negative numbers?
- Are there also non-mathematical remarks (philosophical, historical,…)? Is the difference between plus and minus once as arithmetic operators, once as algebraic symbols clear?
- Applications: How are negative numbers used in calculations? (subtraction, multiplication in algebra, quadratic equations)
- According to your experience, are there parallels or differences to nowadays approaches?
- Other impressions

This workshop was based on parts of the results that we got in the context of the project "*Traditionen der schriftlichen Mathematikvermittlung im 18. Jahrhundert in Deutschland und Frankreich*", financed by the Deutsche Forschungsgemeinschaft (DFG) at the Bergische Universität in Wuppertal. The final aim of this project is to establish a comparison between the German and French textbooks that were used during the 18th century to teach mathematical sciences in higher education. Therefore, we take into account many case studies, including the one at issue. Eventually, we hope to manage to analyze the emergence of traditions in teaching mathematics in this period, and also to retrace their possible origins in the textbooks written in Latin, especially by the Jesuits. To this purpose, we are moreover working on a comprehensive database, based on the software developed by another DFG project, the "*Personendaten-Repositorium*", at the Berlin-Brandenburgische Akademie der Wissenschaften.

FRANCE

All the French authors that we are taking into account delayed the treatment of negative numbers until the algebra part. Bélidor's approach is in this respect peculiar since he did not deal with elementary arithmetic at all, so that negative numbers are explained right at the beginning of his *Nouveau cours de mathématiques* (Book I). Indeed, he took for granted that his readers were acquainted with calculations with integer and fractional numbers and started, after having stated some basic geometrical definitions (without examples), with calculations with "algebraical quantities", that is with letters that are used as signs to point at non-defined numbers. Bélidor maintained that, when an algebraic quantity is preceded by no sign, that is, neither by + nor by – , he always supposed that it has the sign + and called it "positive quantity". On the other hand, the quantities that are preceded by the sign – are called "negative" (cf. Bélidor 1725, p. 11). He provided $+ab = ab$ and $-ab$ as examples, where he used to denote the algebraical quantity ab referring to the extremes a, b of a geometrical segment. Bélidor provided some interpretative models for negative numbers. First, he interpreted them as possessions and debts (cf. Bélidor 1725, p. 14). Later, he stressed that negative quantities are not "less real" than the positive ones. Indeed, they are opposite quantities, which means that they have contrary effects in calculations (cf. Bélidor 1725, p. 18 and p. 80). Bélidor never clearly stated the difference between plus and minus. Sometimes he used them as arithmetic operators, sometimes as algebraic signs. He suggested both viewpoints (cf. respectively Bélidor 1725, p. 8 and pp. 12-13 and Bélidor 1725, p. 14 and p. 18), but he never critically compared them. Negative quantities appear at first while dealing with the algebraical subtraction, where a "$-b$" stands alone. This means that there the "–" denotes the fact that "b" is negative, and it is not an operation. Many other examples, for instance the results of the multiplication $(-8abc)(-5bcd)$ and of the multiplication $(a-b)(a-b)$, can be found in the paragraph on algebraical multiplication. In this passage, Bélidor argued that, if the multiplicand has the sign + (respectively –), the multiplication is made by addition (respectively subtraction) of the same algebraical quantity. A classical example concerns the so-called rule of signs, namely when one or more negative multiplicands are involved. The most interesting examples, however, are to be searched for in the treatment of quadratic equations (cf. Bélidor 1725, pp. 158-166). Indeed, here he provided no general method for solving them, but rather a collection of solved examples. While commenting some of these, Bélidor affirmed that a negative root is to be considered a solution of the problem as well as and with the same degree of trustworthiness as a positive one. Again, he states that negative roots give a solution "in the sense that we intended", meaning that when one finds a negative solution he only has to adapt his interpretation, for instance in terms of debts. Finally, he remarked that the algebraic values are true and reasoned, even if sometimes it seems they don't have a meaning since they are far from what we had imagined.

In contrast, La Caille, as all the other authors that we are taking into account, firstly dealt with arithmetic, then with algebra in his *Leçons élémentaires de mathématiques*.

Again, as in all other works, only in the algebra part do the negative numbers occur. For La Caille algebra is a kind of arithmetic which is more general, faster, briefer, simpler, and that can be applied. Among its preliminary notions, he passed from the definition of “algebraic quantity” quite immediately to the one of “polynomial” (namely, an algebraic quantity that contains more than one term). Here we can find the only definition that can be assimilated to negative numbers: La Caille explained that there are two kinds of terms, the positive ones and the negative ones. These last are always preceded by the sign –, the other by the sign + (cf. La Caille 1741-1750, Vol. 1, p. 62). He only gave the example $+p-q-rr+x-y$, where no term stands alone with a “–”. La Caille interpreted negative numbers as “opposite” quantities and he justified this term in the following way. Indeed, he explained that $-3a$ is the same quantity a taken three times, as for $+3a$, the only difference being that it is taken in the contrary direction. Apart from this and the usual signs rule for multiplication, it is hard to find some other concrete examples. But obviously La Caille is compelled to deal with negative numbers in solving quadratic equations. While giving the general solving method with the quadratic formula, La Caille repeatedly remarked that a solution can be negative (cf. La Caille 1741-1750, Vol. 1, pp. 130-135). He even mentioned that square roots of negative numbers can appear. To this purpose, he limited to explain that it is impossible to find a quantity that, being multiplied with itself, gives a negative product, but to this he added no judgment of value. When the problem that leads to an equation with a negative solution is interpreted in “real” life (for instance, when we search for the number of travelers), a negative solution only points to the fact that also this negative number (for instance, –6) satisfies the equation (cf. La Caille 1741-1750, Vol. 1, p. 135). La Caille also added that “of course” only the positive solution is the one that we were searching for. Further on, La Caille remarked that, when the result of a calculation gives a negative value for the unknown, this means that one has to take this unknown in the opposite direction compared to the one that they considered at the beginning (cf. La Caille 1741-1750, Vol. 1, p. 291).

In Camus' *Cours* negative numbers do not appear. Indeed, it only reaches an elementary level. As all the other French authors, Camus considered only positive numbers in the arithmetic, and algebra is not included at all in the table of contents.

Bézout's treatment of negative numbers in his *Cours de mathématiques* is highly detailed. We take into account the textbook for the navy since, concerning the topic of negative numbers, the differences from the textbook for the artillery are minor. Bézout gave the definition at the beginning of the algebra volume: as usual, the quantities which are preceded by the symbol + are positive, while the ones that are preceded by the symbol – are negative (cf. Bézout 1764-1769, Vol. 3, p. 9). No example is given at first, but then Bézout devotes a whole paragraph to the topic (cf. Bézout 1764-1769, Vol. 3, Réflexions sur les quantités positives et les quantités négatives, pp. 78-84). Among the French authors of our selection, he is the only one that explicitly discusses the distinction of + and – as operations and as properties of quantities. Bézout had already dealt in the usual way with + and – as addition and subtraction in the preceding paragraphs of the arithmetic and algebra parts dedicated to these topics. In this paragraph, he focused on + and – as "the way of being of quantities, one in regard to the others". On the one hand, Bézout legitimated the negative quantities by means of the usual interpretative models, while, on the other hand, he weakened the ontological status of these quantities according to the following arguments. The discussion begins by observing that one quantity can be considered from two opposed viewpoints, and the analogies concerning possessions and debts, and opposite directions on a line are offered to the readers. In this theoretical part (since no examples are provided), Bézout stressed that the negative quantities are as much real as the positive ones except that they have a completely opposite "meaning" in calculations: indeed, negative quantities have properties opposite to the positive quantities, or they behave in an opposite way. On the other hand, when it comes to the applications (in particular in quadratic equations), Bézout provides a conceptual frame to let the students deal with negative numbers. His strategy is to weaken the rights of negative quantities to appear in the solution of a problem. Indeed, Bézout stated that each negative quantity points at a false assumption in the statement of the problem but, at the same time, it also points at its correction, since it would be enough to take the assumed quantity with the opposite symbol.

Finally, Bossut starts the discussion in the algebra volume of his *Cours de mathématiques* by defining, among others, the symbols + and – as operations. The first negative quantity *–b* appears before the definition. According to Bossut, negative and positive quantities are of a same kind, but they are opposite regarding "their way of being" (cf. Bossut 1772-1775, Vol. 2, p. 10). He instantiated this definition with two examples from real life which provide as many interpretations. They boil down as usual to possessions and debts and to considering the opposite direction on a line. In the main, Bossut's textbook shows a lot of similarities with Bézout's one and, in the practice, for instance while dealing with quadratic equations, negative solutions are accepted without reserves. At this point, Bossut did not even need to extensively justify the negative solution. Referring to a numerical equation with a positive and a negative solution, Bossut briefly mentions that both solve the equation (cf. Bossut 1772-1775, Vol. 2, p. 189). His justification is the algebraic calculation in which he simply substituted the two solutions in the equation at issue. In the collection of examples that follows, when the equation derives from a problem with an interpretation in real life, the negative solution (if there is one) is also briefly interpreted as the opposite of the positive one (to gather or to loose water).

GERMANY

Wolff explained the negative numbers within the algebra chapter in the fourth volume of his *Anfangs=Gründe*, namely for solving equations. Wolff did not use the terms "positive", "negative", or "opposed" quantities, but described these quantities as money, debts, and lack (cf. Wolff 1775, Vol. 4, p. 1557).

Kästner is the first author who gave a concrete definition on opposed quantities within the arithmetic chapter at the beginning of his *Anfangsgründe*. Euler treated these numbers in his textbook on algebra. Kästner gave a definition of the opposed quantities:

> Opposed quantities are called quantities from the same kind, which are considered under such conditions that one of them reduces the other one. For instance assets and debts, moving forward and backward. One of these quantities, no matter which one, is called positive or affirmative; the opposed quantity negative or negating (Kästner 1800, p. 71).[2]

Euler defined negative numbers:

> All these numbers, whether positive or negative, have the known appellation of whole numbers, or integers, which consequently are either greater or less than nothing (Euler 1822, p. 5).

Euler proceeded the definition of negative numbers by attribution to a concrete number range, namely the integers. In Euler's definition, there is another aspect which is quite interesting. This concerns the expression "less than nothing". In the 18th century, an unanswered question was the interpretation of negative numbers. From a philosophical point of view, it is very difficult to label negative numbers as "less than nothing", because they are real objects, for instance debts. Therefore, Kästner saw the need to explain the expression "less than nothing" in his textbook (cf. Kästner 1800, pp. 72-74). He stated that one must distinguish between an "absolute nothing" and a "relative nothing". Concerning to the negative numbers, one must choose the meaning of the relative nothing, because a negative number or quantity can only exist because of its opposed (positive) quantity. It is wrong to denote a number negative in an absolute meaning. Euler equated "nothing" with the number "zero" and, with the help of a number line, showed the positive and negative numbers (cf. Euler 1822, p. 5).

During this time, Kästner's definition of the concept of negative numbers was well accepted to be precise. Kästner devoted a whole paragraph (§ 95) to the nature of negative numbers. He impressed with his remarks on the nature of negative numbers and the issue of "less than nothing" even the philosopher Immanuel Kant (1724-1804), who wrote about negative numbers in his work *Versuch den Begriff der negativen Größen in die Weltweisheit einzuführen* (1763).

Wolff explained the negative quantities in order to solve algebraic equations. But we cannot find a lot of examples with references to everyday life. Kästner introduced the negative numbers in a practical way. At the beginning, there are some examples with reference to everyday life. Then you can find "questions" which are used to explain the four basic operations with negative numbers. Kästner uses concrete numbers instead of letters as we can find in Wolff's algebra chapter. Also Euler uses concrete numbers for his explanation of negative numbers. While Wolff only mentions "quantities", Kästner once and Euler several times speak of "numbers". Another observation is that Wolff treats the negative numbers subordinated, while this topic is an independent one in the textbooks by Kästner and Euler.

For the German part, there was a common notion of the interpretation of negative numbers, namely as debts. This is the same as in earlier times (see introduction). Also nowadays this example is very popular and often used for the explanation of negative numbers.

In his book, Euler clearly points out the difference between arithmetic operations and algebraic symbols of plus and minus. First, he explains the arithmetic operations. After that, he introduces plus and minus as algebraic symbols serving as description of positive and negative numbers. Kästner and Wolff did not make this difference clear. This was a problem which Spaun criticized in his writing (see above).

By making reference to the German textbooks, we can see the development concerning the treatment of the negative numbers during the 18^{th} century. This topic was detached from its treatment in the context of algebraic equations. Negative numbers became an independent part, either within the arithmetic or in the algebra chapter. This comes along with the fact that the authors gave a concrete definition of opposed quantities. In order to illustrate negative numbers, Kästner gave a lot of examples from everyday life (like assets and debts). Euler defined the negative numbers as part of the integers and illustrated them at the number line.

CONCLUSION AND SOME RESULTS

There are some interesting observations regarding the treatment of the negative quantities in the considered French and German textbooks. In French textbooks, negative numbers are treated within the algebra part, which is not always the case in Germany. While at the beginning of the 18^{th} century Wolff treated the negative numbers within the algebra part, there is a shift in the course of the years. In the middle of the 18^{th} century, Kästner explained negative numbers at the beginning of his *Anfangsgründe* within the arithmetic part. Although Euler treated the negative numbers in his textbook on algebra, he labeled negative numbers explicitly as “numbers”. Also in Kästner’s textbook we can find once the terminology “number” instead of quantities. On the contrary, the “number”-terminology is never used by the French authors.

There are commonly accepted interpretations of negative numbers, such as possessions and debts, and opposite directions, which are widely employed in both French and German textbooks. Overall, the difference between plus and minus sometimes as operations and sometimes as algebraic properties of quantities is not explicitly addressed; it is completely missing in La Caille since there negative numbers are only defined in the context of polynomials.

The development in France shows a clash of differing epistemological conceptions, which spanned from complete acceptance of negative numbers as solutions of problems (especially when those are originally formulated with no references to real life) to no acceptance (that is, the hypotheses of the problem should be reformulated), passing by a limited acceptance (that is, provided that one can stick to these negative numbers an interpretation to reconnect them with reality).

For the German part we can state that negative numbers were not only regarded as possible solutions in algebraic equations any more. Opposed numbers became an independent topic within arithmetic. This development shows that negative numbers were generally accepted.

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NOTES

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1 Whatever "complete" means depends not only on each single author, but also on the time span. Indeed, there were some shifts in the 18th century in Germany concerning the framework of the mathematical sciences.

2 Translated by Desirée Kröger. Original quote in Kästner 1800, p. 71: "Entgegengesetzte Grössen heissen Grössen von einer Art, die unter solchen Bedingungen betrachtet werden, daß die eine die andere vermindert. Z.E. Vermögen und Schulden, Vorwärtsgehen und Rückwärtsgehen. Eine von diesen Grössen, welche man will, heisst man positiv oder bejahend, die ihr entgegengesetzte negativ oder verneinend".