UNDERSTANDING MATHEMATICS USING ORIGINAL SOURCES. CRITERIA AND CONDITIONS

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ABSTRACT

As an explicit resource in the classroom, the History of Mathematics allows the improvement of the learning of mathematics. Through the analysis of significant texts from the historical evolution of mathematical concepts, the Group of History of Mathematics of the Association of Barcelona for the Study and Learning of Mathematics (ABEAM) has been developing historical materials for use in the classroom since 1998.

In this article we describe the situation in the Catalan curriculum of secondary school mathematics and analyze an example put into practice in the classroom. In addition we discuss the criteria for preparing the historical texts to be used in the classroom, as well as the conditions for using them as a powerful tool for understanding mathematics.

1 Introduction

We would like to recall the words of the Catalan mathematician Pere Puig Adam (1900-1960) on the importance of knowing the history of the concepts,

> "Science grows through a combined process of analysis and a synthesis of induction and deduction. Experience and observations are accumulated, facts recorded, analogies examined, concepts extracted, laws inductively developed and systems deductively constructed. The synthetic presentation of such an elaborate fabric undoubtedly provides solidity to the whole, but it fails to teach how such a fabric is woven, which is what education is about. Thus when used as an instrument for conveying knowledge, these factors have tended to accentuate the separation between two processes that should never have been separated: firstly, the process of the genesis of knowledge and secondly the process of its transmission".[1]

In accordance with this idea, we have prepared and experimented with some materials in the mathematics classroom during some courses. The aim of this article is to discuss through the analysis of an example implemented in the mathematics classroom the criteria and the conditions for transforming these activities that involved historical texts into a powerful tool for learning mathematics. First we present a survey of the evolution of this implementation the last few years in Catalonia.

In Spain, every autonomous community is in charge of their own secondary and graduate education, so we only focus on the implementation of history in the mathematics

[1]La ciencia crece por procesos mixtos de análisis y síntesis de inducción y deducción. Se acumulan experiencias y observaciones, se registran hechos, se examinan analogías, se abstraen conceptos, se inducen leyes y se tejen deductivamente sistemas. La presentación sintética de tales urdimbres da una indudable solidez al conjunto presentado; pero no enseña precisamente a urdir, que es lo educativo. Así, pues, usados como instrumento de transmisión de conocimientos han tendido a acentuar cada vez más la separación entre dos procesos que no debieron divorciarse nunca: el de la génesis de los conocimientos y el de su transmisión (Puig Adam, 1960, p. 95).

classroom in Catalonia.

The implementation of the history of mathematics has for twenty years inspired some individual actions between teachers. Thus, since the academic year 1990/91 financial aid for research work, granted every year to teachers by the Department of Education of the government of Catalonia, has been devoted to research into the relations between the history of science (including mathematics) and its teaching. This research work has resulted in the drawing up of memories that are now available to other teachers. Also in the high schools, workshops, centenaries, and conferences by teachers constitute further examples of activities in which history can be used to achieve a more comprehensive learning experience for students. For example, the workshop devoted to the study of the life and work of René Descartes (1596-1650), held in 1996 at the INS Carles Riba (a Catalan high school), provided students with additional background from a mathematical, philosophical, physical and historical perspective.

As a collective action we may mention that since 2003 to the present, every year Pere Grapi and Mª Rosa Massa have coordinated a workshop on the History of Science and Teaching organized by the Catalan Society of History of Science and Technology (SCHCT), and subsequently they have coordinated the publication of the proceedings. The aim of these workshops is to enable teachers to show their experiences in the classroom as well as to discuss the criteria and conditions for these implementations.

In the academic year 2007-2008, the Department of Education of the government of Catalonia introduced some contents of the history of science into the curriculum for secondary education, namely, the new Catalan mathematics curriculum for secondary schools, published in June 2007, contains notions of the historical genesis of relevant subjects into the syllabus.[2]

In the academic year 2009-2010 a new course has begun for training future teachers of mathematics. The syllabus of this Master's degree launched at the University includes a part on history of mathematics and its use in the classroom. For example, in Polytechnic University of Catalonia the title is: "Elements of history of mathematics for the classroom" and in Pompeu Fabra University the title is: "The history of mathematics and its use to teach math". Also in the academic year 2009-2010, an online pilot course on the history of science for science teacher training was put into practice. This course was produced by historians of science of the Catalan Society for the History of Science and Technology (SCHCT) under the name "Science and Technology through History"

The setting up of the Group of History of Mathematics of Barcelona (ABEAM) in 1998[3] was also a significant step. The aim of this group of teachers of Mathematics is to create History of Mathematics materials to be used in the classroom. The list of the texts implemented includes: *On the sizes and distances of the Sun and Moon* by Aristarchus of Samos (ca. 310-230 BC) (Massa Esteve, 2005b; Aristarco, 2007); Euclid's *Elements* (300

[2]The list of these historical contexts includes: The origins of the numeration system; the introduction of zero and the systems of positional numeration; geometry in ancient civilizations (Egypt, Babylonia); the first approaches to the number π (Egypt, China and Greece); Pythagoras theorem in Euclid's *Elements* and in China; the origins of symbolic algebra (Arab world, Renaissance); the relationship between geometry and algebra and the introduction of Cartesian coordinates; the geometric resolution of equations (Greece, India, Arab World); the use of geometry to measure the distance Earth - Sun and Earth - Moon (Greece).

[3]The coordinator of the group is Mª Rosa Massa Esteve and the other members of group are**:** Mª Àngels Casals Puit (INS Joan Corominas), Iolanda Guevara Casanova (INS Badalona VII), Paco Moreno Rigall (INS XXV Olimpíada), Carles Puig Pla (UPC) and Fàtima Romero Vallhonesta (Inspecció d'Educació). The group is subsidized by the *Institute of Science of Education* (ICE) of the University of Barcelona.

BC) (Romero, Guevara, Massa, 2007); Menelaus' *Spheriques* (ca. 100) (Guevara, Massa, Romero, 2008a-2008b); *Almagest* by Ptolemy (ca. 85-165) (Romero, Massa, 2003); *The nine Chapters on the Mathematical Art* (s. I. AC) (Romero *et al.*, 2009); *Traité du quadrilatère* by Nassir-al-Tusi (1201-1274) (Romero, Massa, Casals, 2006) and *Triangulis Omnimodis* by Regiomontanus (1436-1476) (Guevara, Massa, 2005). The essential ideas on the implementation of history in the mathematics classroom of this group are reflected in the following sections.

2 Teaching mathematics using its history

The History of Mathematics in the mathematics classroom can be used in two ways: as an integral educational resource and as a didactic resource for understanding mathematics (Jahnke *et al.*, 1996; Barbin, 2000; Massa Esteve, 2003-2010; Dematté, 2006;).

In the first sense, history in the mathematics classroom can provide students with a conception of mathematics as a useful, dynamic, human, interdisciplinary and heuristic science.

- A useful science. Teachers should explain to students that mathematics has always been an essential tool in the development of different civilizations. It has been used since antiquity for solving problems of counting, for understanding the movements of the stars and for establishing a calendar. In this regard, there are many examples right down to the present day in which mathematics has proved to be fundamental in spheres as diverse as computer science, economics, biology, and in the building of models for explaining physical phenomena in the field of applied science, to mention just a few of the applications.
- A dynamic science. It will also be necessary to teach students whenever appropriate about problems that remained open in a particular period, how they have evolved and the situation they are in now, as well as showing that research is still being carried out and that changes are constantly taking place. History shows that societies progress as a result of the scientific activity undertaken by successive generations, and that mathematics is a fundamental part of this process.
- A human science. Teachers should reveal to students that behind the theorems and results there are remarkable people. It is not merely a question of recounting anecdotes, but rather that students should know something about the mathematical community; human beings whose work consisted in providing them with the theorems they use so frequently. Mathematics is a science that arises from human activity, and if students are able to see it in this way they will probably perceive it as something more accessible and closer to themselves.
- An interdisciplinary science. Wherever possible, teachers should show the historical connections of mathematics with other sciences (physics, biology, medicine, architecture, etc.) and other human activities (trade, politics, art, religion, etc.). It is also necessary to remember that a great number of important ideas in the development of science and mathematics itself have grown out of this interactive process.
- A heuristic science. Teachers should analyze with students the historical problems that have been solved by different methods, and thereby show them that the effort involved in solving problems has always been an exciting and enriching activity at a personal level. These methods can be used in teaching to encourage students to take an interest in research and to become budding researchers themselves.

In the second sense, the history of mathematics as a didactic resource can provide tools to enable students to grasp mathematical concepts successfully. The History of mathematics can be employed in the mathematics classroom as an implicit and explicit didactic resource.

The history of mathematics as an implicit resource can be employed in the design phase, by choosing contexts, by preparing activities (problems and auxiliary sources) and also by drawing up the teaching syllabus for a concept or an idea.

Nevertheless, it is necessary to bear in mind that the historical process of building up a body of knowledge is a collective task that depends on social factors. In the past, many mathematicians adopted the solution of particular problems as the aim of their research and were able to devote many years to their objectives. It is worth remembering that our students, while having the ground before them well-prepared, are addressing these notions for the very first time and often lack the motivation for solving mathematical problems.

Indeed, it is not history itself that is relevant for teaching, but rather the genesis of problems, the proofs that favored the development of an idea or a concept. The clarification of this development of ideas and notions can also act as a motivation for solving current problems. The evolution of a mathematical concept can thereby reveal the learning difficulties encountered by students, as well as pointing the way towards how the concept can be taught (Massa Esteve, 2005a).

In addition to its importance as an implicit tool for improving the learning of mathematics, the history of mathematics can also be used explicitly in the classroom for the teaching of mathematics. Although by no means an exhaustive list, we may mention four areas where the history of mathematics can be employed explicitly: 1) for proposing and directing research work at baccalaureate level using historical material[4]; 2) for designing and imparting elective subjects involving the history of mathematics; 3) for holding workshops, centenaries and conferences (Massa Esteve, Comas, Granados, 1996), and 4) for using significant historical texts in order to improve understanding of mathematical concepts (Massa Esteve, Romero, 2009). In this article we analyze the last point.

3 Using significant historical texts in the mathematics classroom: Criteria and conditions

The use of significant historical texts in the classroom to facilitate the understanding of mathematical concepts is the activity that can provide students with more valuable means for learning mathematics.

The main aims for its implementation in the mathematics classroom are to achieve that students a) know the original source on which the knowledge of mathematics in the past is based, b) recognize the socio-cultural relations of mathematics with the politics, religion, philosophy or culture in a certain period and, last but not least, c) improve mathematical thinking through the reflections on the development of mathematical thought and the transformations of natural philosophy.

What historical texts are suitable for use in the mathematics classroom? Not all historical texts are useful for the mathematics classroom. Initial selection could be historical texts related to the historical contexts in the new Catalan curriculum. Historical texts (e.g. proof or problems) have to be in some way anchored in the mathematical issue. Different types of

[4] The list of titles of these research works can be very long, for instance: Pythagoras and music, On Fermat's theorem, On Pascal's Arithmetic Triangle, On the beginning of algebraic language, Women and science, On the incommensurability problem, Scientific Revolution,…

historical texts should be used, depending on the step in the didactical sequence.

At what moment in the teaching process should we use historical texts in the mathematics classroom? Historical texts could be used to introduce a subject or a concept, to explore it more deeply, to explain the differences between two contexts, to motivate study of a particular type of problem or to clarify a process of reasoning.

How do we use historical texts in the mathematics classroom? We must keep in mind some points: It is necessary to clarify the relation between the historical text and the mathematical concept under study, so that the analysis of the text or significant proof should be integrated into the mathematical ideas one wishes to convey. The mathematical reasoning behind the proofs should be analyzed. Indeed, addressing the same result from different mathematical perspectives enriches students' knowledge of mathematical understanding. The proof should be contextualized within the mathematical syllabus by associating it with the mathematical ideas studied in the course so that students may see clearly that it forms an integral part of a body of knowledge, and it should also be situated within the history of mathematics to enable students to evaluate the historical development of the concept.

In order to use historical texts properly, teachers are required to present some features of the historical period and also to describe historical figures in context, both in terms of their own objectives and the concerns of their period. Situating authors chronologically enables us to enrich the training of students by showing them the different aspects of the science and culture of the period in question in an interdisciplinary way. It is important not to fall into the trap of the amusing anecdote or the biographical detail without any historical relevance. It is also a good idea to have a map available in the classroom to situate the text both geographically and historically.

As an example, we now provide a description of the implementation process of a significant historical text in the mathematics classroom.

3.1 Aristarchus of Samos in the mathematics classroom

The activity we are going to present deals with the work *On the Sizes and Distances of the Sun and Moon* (ca. 287 BC) by Aristarchus of Samos (ca. 310-230 BC). In order to implement the activity, we begin with a brief presentation of the epoch, Greek astronomy, and the person of Aristarchus himself. Then we situate his work in the history of trigonometry, analyze the aims of the author as well as the features of the work, and finally students are prompted to follow the reasoning of this work, Proposition 7, in order to arrive at new mathematical ideas and perspectives (see this material in annex). This classroom activity was implemented in the last cycle of compulsory education (14-16 year old).

3.1.1 The context: Greek astronomy

As many historians point out, the history of Greek astronomy probably began as part of the history of Greek philosophy, so that the first great philosophers were also the first astronomers. Thus we can quote, for example, Thales (ca. 624-547 BC), Pythagoras (ca. 572-497 BC), Eudoxus (ca. 408-355 BC), Aristotle (ca. 384-322 BC), etc.

According to Heath, Thales, known as an astronomer, predicted and explained the causes of an eclipse of the Sun; he understood the Moon and the Sun as disks or short cylinders that behaved as if floating in water (Aristarchus, 1981, pp. 137-138).

Other developments took place with Pythagoras and his followers, who recognized that the Earth was a sphere and that Venus, the evening star, was the same planet as Venus, the morning star. The motion of the Earth, the Sun, the Moon and the planets around a central fire was also a theory attributed to a disciple of Pythagoras, Philolaus of Crotona (ca. 470 BC).

Subsequently, Eudoxus proposed a theory of homocentric spheres to describe the motion of celestial bodies. He surmised that the Earth sat motionless at the center and the planets (including the Sun and the Moon) followed a circular course around it.

Aristotle, whose texts had great influence, analyzed observable realities and reconstructed universe theory in cosmology, integrating many of the ideas of his predecessors such as the geocentric, the structural framework of the universe of the two spheres, the Platonic principle of circular motion and even of the heavenly bodies, as well as the pre-Socratic theory of the four elements (Puig-Pla, 1996, pp. 41-55). He laid the groundwork for what we now call old physics and the basic lines of his doctrine were accepted as dogma for about sixty generations. Available sources that refer to the principles of natural philosophy of Aristotle are the eight books of *Physics*. Astronomical issues are discussed mainly in the four books of *De Caelo* and *Meteorology*. In fact almost all Greek, Arab and Christian astronomers accepted, implicitly or not, the fundamental premises of Aristotelian cosmology: the closed and finite nature of the cosmos, the immobility of the earth at the center of the universe and the essential difference between the two regions: the blue (superlunary) and terrestrial (sublunary).

Aristarchus of Samos, who appears between Euclid (ca. 300 BC) and Archimedes (287-212 BC), was one of the rare exceptions and put forward heliocentric ideas of the universe, as discussed later. However, Aristarchus, in his work *On the sizes and distances of the Sun and Moon*, used the geocentric theory and was one of the first to write a work that estimated the sizes of the Sun and Moon in relation to those of Earth and the distances from them to Earth.

3.1.2. The historical author: Aristarchus of Samos.

Aristarchus was born on the island of Samos in 310 BC. He was a pupil of Straton of Lampsacus, the third director of the Lyceum, the school founded by Aristotle. Little is known about his life. The limited information available is determined by the quotations found in later texts and the works he left behind. Thus, Ptolemy, in his *Almagest* (150), also called *Syntaxis Mathematics*, explains that Aristarchus observed the summer solstice in 280 BC. Ptolemy in the first paragraph of Book III of his work also mentions Aristarchus' procedures to determine the length of the solar year (Ptolemy, 1984, pp. 137-139). Later, Nicholas Copernicus (1473-1543) in his *De revolutionibus orbium coelestium* libri VI (1543) explains the observations of Aristarchus at Alexandria with Timocharis of Alexandria (approx. III BC) and Aristyllus (disciple of Timocharis).

Although he was recognized as an astronomer in his earlier works, in his time Aristarchus was called the "mathematical", and is quoted as one of the few men having a thorough knowledge of all branches of science, geometry, astronomy, music, ... For instance, Vitruvius (first century BC) mentions him in his work *De Architectura* (ca. 25 BC). On the other hand, we have no doubt that he was a very able mathematician, as is evident in the work of astronomy that he has bequeathed us. He also wrote about vision, light and colors. He said the colors were "forms imprinting the air with impressions of their own true nature."

However, Aristarchus is mostly known as the "old Copernicus." Historians are unanimous in stating that Aristarchus was the first to introduce the heliocentric hypothesis. Archimedes (Archimède, 1971, p. 135), his contemporary, affirms in a passage of his work *Sandreckoner* (216 BC) that it could be deduced that Aristarchus assumed that the spheres of the stars and the sun remained motionless in space and that the Earth revolved around the Sun. Aristarchus compared the sphere of fixed stars with the orbit of the Earth. In this sense, he is also quoted by Plutarch (ca. 46-125) in his work *Moralia*, which says that Cleanthes thought that Aristarchus should be attacked for moving the Earth to the center of the universe; that is to say, it was assumed that Aristarchus included the motion of the Earth in his theories. However, despite these references, Aristarchus in his work *On the sizes and distances of the Sun and the Moon* does not mention the heliocentric hypothesis. This hypothesis suggests that in his work he recognized that the Sun is much larger than the Earth and Moon and is much farther from Earth than the Moon. We now consider the content of this work in more detail.

3.1.3 The historical work: *On the sizes and distances of the Sun and Moon*.

This work according to Pappus was found in a collection of texts called *Small Astronomy* together with other works like: *On the sphere in mouvement* by Autólico of Pitania (ca. 320 BC), *Optic and Fenomena* by Euclid, *Sphaerics* and *De diebus et noctibus* by Teodosio (ca. 107-43 BC) and others.

From six hypotheses about the sizes and distances to the stars, and by means of eighteen propositions, Aristarchus demonstrates three theses. The contents of the first three hypotheses could be stated thus: the first says that the moon receives its light from the Sun; the second explains that the Earth is the center of the sphere in which the moon moves, and in the third he describes the maximum circle defining the parts of darkness and brightness of the moon that are in the direction of our eye. These scenarios therefore provide no angle and no measure, but rather describe the positions of the stars. The other three assumptions provide measurements probably obtained by observation. Thus, the fourth hypothesis means that when the moon is at right angles to the Sun and the Earth, the viewing angle of the Moon from Earth is 87 degrees, and the other angle of the triangle measured one-thirtieth (1/30) a quadrant (90 degrees), or 3°; the fifth gives that the size of the Earth's shadow is twice the Moon, and the sixth and final hypothesis explains that the moon as seen from Earth forms a cone, with an angle of 2°, which is one fifteenth of a zodiac sign (30 degrees).

The three theses found at the beginning of the book are: first, the distance of the Sun from the Earth is greater than eighteen times, but less than twenty times, the distance of the Moon from the Earth; the second, the diameter of the Sun has the same ratio as the diameter of the Moon, and the third, the Sun's diameter has to the diameter of the Earth a ratio greater than that which 19 has to 3, but less than that which 43 has to 6. The proof of Proposition No. 7, which is the first thesis, is the one we use to design the classroom activity.

Its rigor in reasoning was not accompanied by correct observations, and noted an angle of 87 degrees, when in fact it is almost 90 degrees. In fact Pappus says in his commentary that Aristarchus deduced previous relations, assumptions and comments, and said that changing the assumptions also changed the measurements obtained. So later Hipparchus and Ptolemy deduced other relationships approximating those we know today.

3.1.4 The significant text

A comprehensive evaluation of this work of astronomy must take into consideration the close relationship between the beginnings of astronomy and the origins of trigonometry, an aspect that contributes to a greater understanding. In the text, Aristarchus posed plane geometry problems, cutting the spheres of the Sun and Moon into maximum circles. To solve geometric problems, he relied on relationships, today regarded as trigonometric, between angles and sides of a triangle. The angles are expressed as fractions of the square and he wrote the trigonometric ratios as ratios of sides of triangles, so the upper and lower bounds of the value required can be determined.

Aristarchus' geometrical propositions are mainly based in Euclid's *Elements*. Eudoxus' theory of proportions in Book V of the *Elements* is used consistently, and its properties of inverting, alternating, composing and multiplying are implemented for equal proportions and also for inequality proportions. Aristarchus also bases these propositions implicitly on other relationships, which for us are trigonometric, as if he knew these relations or considered them to be trivial.

The text is a collection of propositions consistent with a sequential description of the ideas one wishes to display, bearing in mind their objectives, i.e., to calculate the sizes and distances of the stars. The propositions are mathematical exercises and operations between ratios and the singular construction of geometrical figures are evidence of the high quality of this mathematician. This is a rich, well structured text, and in our opinion the proofs are flawless in terms of rigor.

3.1.5. The significant proof: the proposition 7

In this activity, students are prompted to reproduce Proposition 7, which deals with the ratios between the distance of the Sun and the Moon from the Earth.

Aristarchus, *Proposition 7*

"The distance of the Sun from the Earth is greater than eighteen times, but less than twenty times the distance of the moon from the Earth" (Aristarchus, 1981, p. 377).

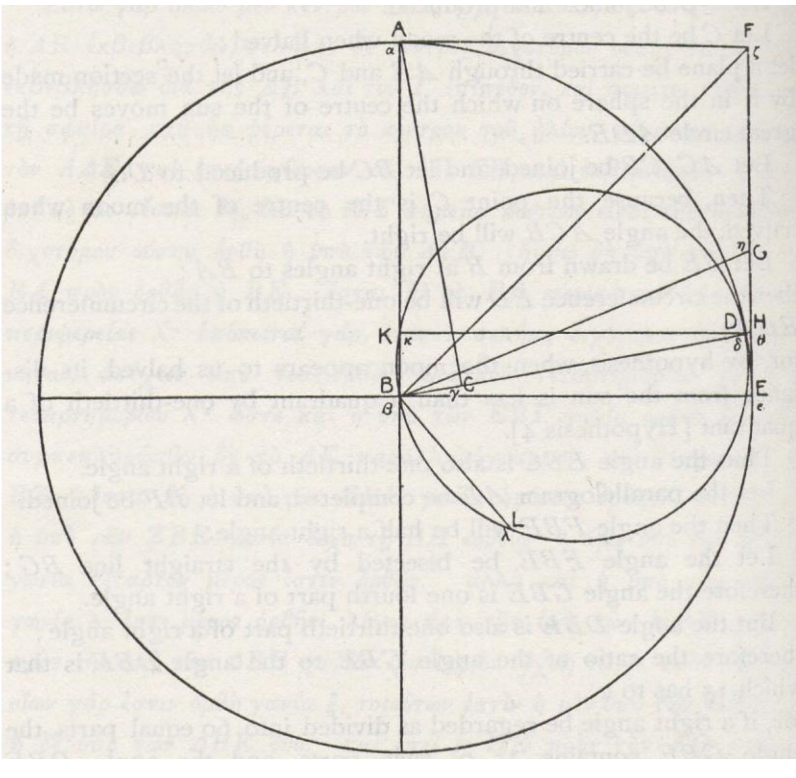

Figure 1. Aristarchus' Illustration (Aristarchus, 1981, p. 378)

In fact, B being the centre of the Earth, A the centre of the Sun and C the centre of the Moon (see Figure 1), Aristarchus showed:

$$1/18 > \sin 3^{\circ} = CB : AB > 1/20.$$

Aristarchus translated the problem from the triangle ABC to the constructed similar triangle BHE. He based his proof on the value of the angle ABC, which he stated as 87º and which he determined by observation (see the comment in the Section 3.1.3).

The angle FBE is 45º and the angle GBE the half, then the ratio between two angles GBE and DBE, which is 3º, is

$$GE : HE > (GBE) : (DBE) = 15 : 2$$

As $FB^2 = 2\ BE^2$, he used proportions in similar triangles and then $FG^2 = 2\ GE^2$. Using the inequality: $50 : 25 = 2 > 49 : 25$. So:

$$FG : GE > 7 : 5 \text{ and } FE : GE > 12 : 5 = 36 : 15.$$

Therefore by composing FE : GE with GE : HE we obtain FE > 18 HE and BE > 18 HE. Then, since the triangles ABC and BHE are similar, AB > 18 CB.

In the development of the implementation of the activity in the classroom we present Aristarchus in the context of Greek astronomy and the historical work: *On the sizes and distances of the Sun and Moon*. Then we analyze the text or significant proof by making a careful assessment of the characteristics and the mathematical reasoning behind the proofs. In this case, together with the students, after the implementation of the dossier (see annex), we should analyze the four mathematical strategies used by Aristarchus in this proof. The translation of the analysis of the triangle Sun-Earth-Moon to a similar triangle, the use of the relationship, as if it were trivial, between the tangents and angles (in actual notation $\text{tg}\,\alpha : \text{tg}\beta > \alpha : \beta$, with angles β, α of the first quadrant), the establishment of a ratio between the segments which determines the angle bisector and the sides of the triangle (using a proposition from the *Elements*), and lastly the approach of $\sqrt{2}$ by $7 : 5$. At the end, students move the result obtained in the similar triangle to the initial triangle ABC, Sun-Earth-Moon and conclude that AB > 18 CB.

4 Remarks

We have designed activities related to several topics, in geometry, trigonometry and algebra, which may also contribute to improving the students' mathematical education.

According to the criteria and conditions mentioned above, we have created new teaching materials containing explanations that use a constructive learning method. This implementation in the mathematics classroom is designed for students to follow their own reasoning, just like the old mathematics did. The analysis of significant proofs reveals the different ways those students have of working and approaching problems, thereby enabling them to tackle new problems and to develop their mathematical thinking. The analysis of historical texts improves the students' overall formation, giving them additional knowledge about the social and scientific context of those periods.

Thus we may conclude that the use of the history of mathematics in our teacher training courses, both at the initial and permanent stages, should extend the knowledge of teachers, should enrich the learning of students and should improve the quality of mathematical training as a whole.

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On-line course: http://www.xtec.cat/formaciotic/dvdformacio/materials/tdcdec/index.html

ANNEX

DOSSIER ALUMNES ARISTARC

SOBRE LES MIDES I LES DISTÀNCIES DEL SOL I LA LLUNA (260 aC)

En aquesta obra, Aristarc de Samos (310 aC- 230 aC) parteix de sis hipòtesis sobre les mides i distàncies als astres i mitjançant divuit proposicions demostra tres tesis. Anticipant-se als mètodes trigonomètrics posteriors Aristarc va ser el primer en desenvolupar procediments geomètrics per aproximar els sinus d'angles petits.

Les hipòtesis en les que es basa són:

1.- La Lluna rep la llum del Sol.

2. - La Terra és com un punt al centre de l'esfera en la qual es mou la Lluna.

3.- Quan la Lluna se'ns mostra partida en dues parts, el gran cercle que separa la foscor i la claror de la Lluna s'inclina cap a la nostra visió.

4.- Quan la Lluna se'ns mostra partida per la meitat, llavors la mateixa Lluna s'allunya del Sol menys d'una quarta part (90º) en 1/30 part d'un quadrant (o sigui en 3º).

5.- L'amplada de l'ombra de la Terra es suposa com dues Llunes.

6. - La Lluna subtendeix una quinzena part d'un signe del zodíac (o sigui 1/15 part de 30º)

Diu Aristarc que amb aquestes hipòtesis pot provar la Proposició 7

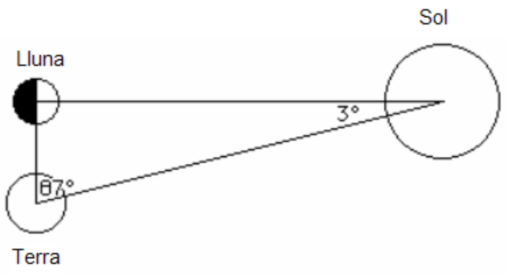
La distància al Sol des de la Terra és més gran que divuit vegades, però més petita que vint vegades, la distància a la Lluna des de la Terra.

És a dir:

18 · distància Terra-Lluna < distància Terra-Sol < 20 · distància Terra-Lluna

Demostrarem la primera desigualtat :

distància Terra-Sol >18 · distància Terra-Lluna

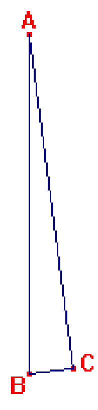


Demostració:

Sigui A el centre del Sol, B el centre de la Terra, i C el centre de la Lluna quan se'ns mostra partida per la meitat, llavors BC representa la distància Terra-Lluna i BA representa la distància Terra-Sol.

Reproduirem la demostració d'Aristarc pas a pas, construint progressivament el dibuix que ell mateix va utilitzar.

Considerem el triangle:



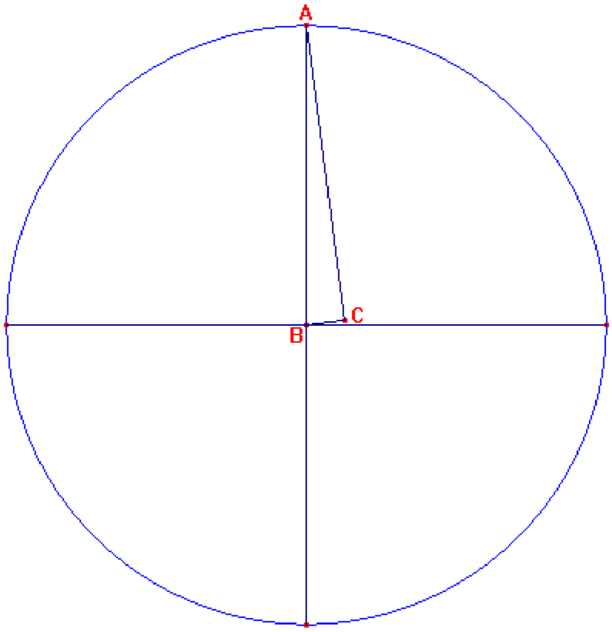
en el que A representa el Sol, B la Terra i C la Lluna quan se'ns mostra partida per la meitat.

El que volem demostrar és que BA >18·BC

Segons la hipòtesi 4, l'angle CAB és de 3º i l'angle BCA és recte.

Per tant l'angle ABC que mesura l'allunyament de la Lluna al Sol deduïm que és de …………

Construïu ara sobre el triangle següent la circumferència amb centre B i radi BA, prolongueu aquest radi per obtenir un diàmetre i traceu també el diàmetre perpendicular a aquest.



Anomeneu E l'extrem dret del diàmetre perpendicular al BA. Pel punt E aixequeu una perpendicular, prolongueu el costat BC i anomeneu H la intersecció d'aquestes darreres rectes.

Fixeu-vos ara en el triangle BEH que acabeu de construir i indiqueu el valor dels angles:

HBE: . . . . .

BEH: . . . . .

EHB: . . . .

Com són els triangles ACB i BEH? . . . .

Tot seguit Aristarc estudia el problema en aquest nou triangle BEH semblant a l'anterior i demostrarà que BH >18 · EH (1)

Completeu el dibuix construint el quadrat determinat pels costats AB i BE i anomeneu F el quart vèrtex. Traceu la diagonal BF del quadrat, llavors l'angle FBE val 45º. Traceu la bisectriu de l'angle FBE que tallarà al quadrat en G, llavors l'angle GBE val la meitat o sigui 90/4.

Fent la raó entre els dos angles GBE i HBE, s'obté: $\frac{GBE}{HBE} = \frac{90/4}{3}$ i simplificant: …. . .

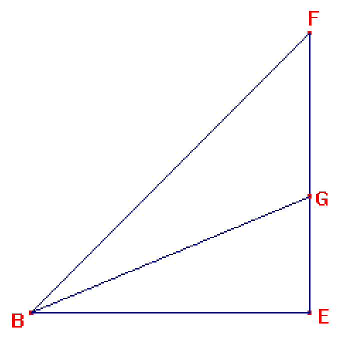
Diu Aristarc que, com que sabem que la raó entre els costats oposats a aquests angles és més gran que la raó entre ells, podem escriure que :

GE : HE > (GBE) : (HBE) = 15 : 2 (2)

Ara, aplicant el teorema més conegut referit als triangles rectangles, ... , *(indiqueu el nom d'aquest teorema)* al triangle rectangle isòsceles BEF, obtenim:

$$FB^2 = 2\,BE^2$$

Considereu ara el triangle BEF amb la bisectriu BG de l'angle FBE:



Traceu, des de G la perpendicular al costat BF i anomeneu E' el peu d'aquesta perpendicular. El triangle FE'G és rectangle i isòsceles.
Perquè l'angle GE'F és . . . . i els angles GFE' i E'GF són tots dos de . . . . .
En aquest triangle isòsceles apliquem el teorema de Pitàgores, llavors

$$FG^2 = 2GE'^2$$

A més a més els dos triangles BEG i BE'G són iguals perquè tenen un costat comú . . . . i els tres angles iguals. O sigui que GE'=.... En aquestes condicions

$$FG^2 = 2GE^2$$

Aquesta darrera igualtat, llegida en forma de proporció diu: $FG^2 : GE^2 = 2$
i d'altra banda 2 = 50 : 25 > 49 : 25. Aristarc escriu $FG^2 : GE^2 > 49 : 25$. Tria aquests dos valors, 49 i 25 perquè són quadrats perfectes i traient l'arrel quadrada queda

FG : GE > 7 : 5.

D'altra banda com que FE = FG + GE , en lloc de treballar amb FG podem fer-ho amb FE, component la raó:
FE : GE = (FG + GE) : GE = FG: GE + GE : GE > 7 : 5 + 5 : 5 = 12 : 5 = 36 : 15
és a dir:

FE : GE > 36 : 15

Recuperem de (2) la raó GE : HE > 15 : 2 . Multiplicant-la per l'anterior obtenim

FE : HE > . . . .

És a dir que FE > 18 HE
Retornem ara al quadrat ABEF. FE = BE, i per tant, podem escriure BE > . . . . .
Com que BH és la hipotenusa i BE és un catet del triangle BEH, BH . . . . BE
i, per tant, ... i queda demostrada la desigualtat (1).
Per acabar, recordem que el triangle BEH era una construcció addicional per estudiar les desigualtats trobades però el nostre triangle original Sol-Terra-Lluna era BC.
Quina relació hi havia entre el triangle ACB i el BEH? . . . .
Quin seria el costat corresponent al BE en el triangle ACB? . . . .
Quin seria el costat corresponent al HE en el triangle ACB? . . . .
I ara la conclusió final.
En aquestes condicions la desigualtat BH > 18 HE del triangle BEH implica la desigualtat . . . . en el triangle ACB
Escriviu una frase final que, utilitzant la darrera desigualtat, expliqui la relació entre la distància del Sol a la Terra i la distància de la Lluna a la Terra
. . . .