

HISTORICAL CONFLICTS AND SUBTLETIES WITH THE $\sqrt{\quad}$ SIGN IN TEXTBOOKS

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ABSTRACT

The ambiguity of the square root sign is a problem with historical origins. In this paper some of these historic origins are traced and discussed so as to enable us to give an account of the problem in two dimensions: current mathematical developments and the teaching proposal as it is shown in textbooks today. In conclusion, it can be said that although the problem does not exist from the point of view of current developments in mathematics, one cannot say the same thing from the point of view of mathematical education. On the contrary, in our study we have identified conceptual and operative approaches, and also omissions, with the $\sqrt{\quad}$ sign, in modern and representative Spanish textbooks that may be the cause of deeply-rooted misunderstandings and conflicts.

1 Introduction

The ambiguity of the square root is a problem with historical origins, as we can see in important and influential textbooks published at about the time the school system began to be reorganized into a general system of education, with repercussions as regards the way of organizing present-day elementary school teaching of algebra.

Although the problem does not exist from the point of view of present-day developments in mathematics, one cannot say the same thing from the point of view of mathematical education.

On the contrary, the learning of the square root and the sense of use of the radical sign, $\sqrt{\quad}$, present conflicts and subtleties as a consequence of the change in the sign's meaning when passing from arithmetic to algebra, and when going on to work with positive to negative numbers. This often goes unnoticed by teachers and textbooks.

This lack of perception may be the cause of conceptual and operative misunderstandings experienced by teachers and students. For example, students' opinion about the statement $\sqrt{25} = \pm 25$ (Roach, Gibson & Weber, 2004), or $(-8)^{1/3} = -2$ (Goel & Robillard, 1997; Tirosh & Even, 1997), or the rule for multiplying square roots $\sqrt{a} \times \sqrt{b}$ when a and b are negative numbers: $\sqrt{-4} \times \sqrt{-9} = -6 \neq \sqrt{-4 \times -9} = 6$ (Martínez, 2007), and the conflict with equality $\sqrt[6]{3^2} = \sqrt[3]{3}$ because on the left the index of the root is even, so that it has two opposing roots, whereas on the right the index is odd so it only one root (Gómez & Buhlea, 2009; Necula & Gómez 2009).

One way to give an account of the problem is "going back to history to study the evolution of algebraic ideas, analyzing historical texts as cognitions in the same way that we analyze students' productions, which in turn constitute mathematical texts" (Gallardo, 2008).

2 Historical traces of ambiguity in the square root operation and in the radical sign $\sqrt{\quad}$

There are examples of ambiguity in the square root to be found in the teaching tradition that appears in such influential textbooks as like Euler's *Elements of algebra* (1822),

Peacock's *Treatise on Algebra* (1845), De Morgan's *Elements of algebra* (1837) and Lacroix's *Elements of Algebra* (1831). Euler considered that "the square root of any number always has two values, one positive and the other negative; that $\sqrt{4}$, for example, is both $+2$ and -2 , and that, in general, we may take $-\sqrt{a}$ as well as $+\sqrt{a}$ for the square root of a " (Euler, 1822, p. 44).

In this text of Euler's, the square root operation and the radical symbol are ambiguous in their value, because they have two numerical values, in this case $+2$ and -2 . Not only that, but the radical symbol $\sqrt{\quad}$ is used with a duality of meaning and sense of use, which is different when it goes with a number $\sqrt{4}$ or with a letter \sqrt{a} . In $\sqrt{4}$ it is associated with the set of two numbers: ± 2 and in $+\sqrt{a}$ it is associated only with one number, an absolute value susceptible to the $+$ or $-$ sign. And in $\sqrt{4}$ it is used in order to indicate an operation in an abbreviated way, but in $+\sqrt{a}$ it is used in order to express the result of this operation. In short:

1. Ambiguity of the operation
 - The square root of 4 is not one number, but two: $+2$ and -2 .
 - $\sqrt{4}$ has two numerical values: ± 2 .
2. Duality of meaning of the symbol $\sqrt{\quad}$
 - $\sqrt{4}$ is associated with the set of two numbers: ± 2 .
 - \sqrt{a} is associated only with one number, an absolute value susceptible to the $+$ or $-$ sign.
3. Duality in the sense of use of the symbol $\sqrt{\quad}$:
 - $\sqrt{4}$ is used to indicate an operation.
 - \sqrt{a} is used to express the result of the operation.

On moving from arithmetic to algebra, the signs take on new sense in use

As we can see in Euler's text, on moving from arithmetic to algebra the square root symbol acquires new meaning and sense in use. In arithmetic, the square root of 4 can be found and it is unique, 2, which is written $\sqrt{4}$. Things change in algebra, since the square root of a cannot be calculated, so that to indicate its value the expression \sqrt{a} is introduced, which not only represents an indicated operation but a result¹.

Traces of this change of sense in use can be found, for example, in the *Summa* (1494) by Pacioli who indicates the addition and subtraction of quantities of the same nature as $3x$ and $4x$ with words from the vernacular language: "with" and "from". On the other hand, addition and subtraction of non-homogenous quantities such as $3x$ and $4x^2$ are expressed with a symbol "p" or "m", which not only indicate the operation but also its result: "4co with 3co we shall say make 7co, and ... 3co from 7co we shall say subtract 4co, because they are of the same nature ... if we wish to know 3co with 4ce, we shall say that they are 3co p 4ce" (op. cit. *Distinctio octava. Tractatus Primus*, p. 112).

Traces of conflicts with ambiguity in the square root operation

In the 19th century, when today's bases of mathematics were being set down, many of the

¹ The point of view of the psychology of the mathematics education supports this interpretation with the fact that a mathematical entity can be seen as an object and a process. Treating a mathematical notion as an object leads to a type of conception called structural, whereas interpreting a notion as a process implies a conception called operational (see Kieran (2006) and Sfard (1991), for example).

most influential mathematicians such as Peacock, De Morgan and Lacroix, recognised that the square root operation was ambiguous. For example, Peacock² says: “In passing from the square to the square roots, we shall always find two roots, which only differ from each in their sign” (Peacock, 1845, vol. II. p. 67).

And then, Peacock extends the ambiguity of the square root operation to two equivalent forms of symbolic notation: $a^{1/2}$ and \sqrt{a} . He says: “It follows, therefore, that $(a^2)^{1/2} = a^{2/2} = a = \sqrt{a^2}$ * (*The square root of a^2 may be $-a$ as well as $+a$) (op. cit. p. 67). In this paragraph, Peacock seems to have us understand that: $(x^2)^{1/2} = \sqrt{x^2} = \pm x$.

Unlike Peacock, De Morgan proposes differentiating the two forms of symbolic notation for the square root: \sqrt{a} and $a^{1/2}$: “Having two symbols to indicate the root of a , namely, \sqrt{a} and $a^{1/n}$, we shall employ the first in the simple arithmetical sense, and the second to denote any one of the algebraical roots, that is, any one we please, unless some particular root be specified. Thus $\sqrt{4}$ is 2, without any reference to sign; but $(4)^{1/2}$ may be either $+2$ or -2 ” (De Morgan, 1837, p. 122-123). That is, De Morgan tells us: $(4)^{1/2} = \pm 2$ and $\sqrt{4} = 2$.

On the other hand, Lacroix recommends a general rule for the sign that affects the square root: “The double sign \pm is to be considered as affecting the square root of every quantity whatever” (Lacroix, 1831, p. 122).

This rule is accompanied by a subtle question: “It may be here asked, why x , as it is the square root of x^2 , is not also affected with the double sign \pm ?” (op. cit. p. 122).

This is the same as: why $\sqrt{x^2} = x$ and not $\pm x$? His answer was: “if in resolving the equation $x^2=b$, we write $\pm x = \pm\sqrt{b}$, and arrange these expressions in all the different ways, of which they are capable, namely: $+x = +\sqrt{b}$; $+x = -\sqrt{b}$; $-x = -\sqrt{b}$; $-x = +\sqrt{b}$, we come to no new result, since by transposing all the terms of the equations: $-x = -\sqrt{b}$, $-x = +\sqrt{b}$, or which is the same thing, by changing all the signs, these equations become identical with the first” (op. cit. p. 123).

Two epistemological conceptions emerge from the above. One is mentioned by Euler, where ambiguity affects the value of the radical sign and the square root operation (there is no difference between them and the sign and the operation are interchangeable):

$$\sqrt{4} = \pm 2 \quad \text{and the square root of } 4 = \pm 2$$

The other is seen in De Morgan’s text, where ambiguity does not affect at the value of the radical sign:

$$\sqrt{4} = 2 \quad \text{but } (4^2)^{1/2} = \pm 2$$

Furthermore, a rule is emphasised in order to solve equations of the type $x^2=a$, for which the double sign \pm does not affect the square root of x^2 :

$$x^2 = 9 \rightarrow x = \pm\sqrt{9} \rightarrow x = \pm 3$$

3 A mathematical problem and an educational problem

These “epistemological / historical” conceptions reveal a mathematical problem and an educational problem.

² A *Treatise on Algebra* by George Peacock is the result of Peacock’s desire to draft a text with which his students could make sense of the emergence of the algebraic sign system (Gallardo, 2008)

The mathematical problem is that they violate formal requisites: one is the general requirements for any mathematical definitions and another is the requirements of the definitions of mathematical operations with real numbers (see Even & Tirosh, 1995).

The educational problem is that they create conflicts and misunderstandings in students, teachers, and even in school textbooks. For example, if $\sqrt{a^2} = \pm a$, then

$$\sqrt[6]{3^2} \neq \sqrt[3]{3} \Leftrightarrow 3^{\frac{2}{6}} \neq 3^{\frac{1}{3}}$$

as the index of the first radicand is an even number, two solutions exist (one being the opposite of the other) but in the second case, the index is an odd number and therefore there is a single root (see this conflict in the case of Patricia, a high school mathematics teacher in Gómez and Buhlea 2009, or Necula and Gómez, 2009).

Consequently, it can be said that $\sqrt[n]{a^m} \neq \sqrt[kn]{a^{km}}$, when kn is even and n is odd. But, this statement contradicts the definition of the rational exponent: a^r , $r \in \mathcal{Q}$, since this must not depend on the representatives of the numbers involved in the operation.

$$\text{If } r = \frac{m}{n} = \frac{km}{kn} \text{ then it must follow that } a^r = a^{\frac{m}{n}} = \sqrt[n]{a^m} = a^{\frac{km}{kn}} = \sqrt[kn]{a^{km}}$$

Furthermore, $\sqrt{x^2} = \pm x$ violates the general requirements of the definitions of mathematical operations on real numbers. The basic arithmetic operations of addition and multiplication by a number different from zero establish univalent functions³:

$$x \rightarrow x + a, \quad x \rightarrow x \cdot a, \quad a \neq 0.$$

These functions have unique inverse functions corresponding to the inverse operations. However, the operation: $x \rightarrow x^2$ does not establish an univalent function; because $x^2 = (-x)^2$, and therefore it does not have an inverse function⁴. So, the operation: $x \rightarrow$ square root of x does not establish a function.

In order to solve the problem of the ambiguity of the radical symbol, mathematicians have decided to assign to the expression \sqrt{x} , $x \geq 0$ only one value, one of the roots of x , the non-negative root, the one that they name principal root⁵. With this restriction, the correct thing is to write $\sqrt{4} = 2$, not ± 2 .

But this does not signify that $\sqrt{x^2} = x$, because this gives rise to incoherencies that usually go unnoticed: $\sqrt{x^2} = x \rightarrow -2 = \sqrt{(-2)^2} = \sqrt{4} = \sqrt{(+2)^2} = +2$. The correct value from the point of view of current mathematics is $\sqrt{x^2} = |x|$, not $\sqrt{x^2} = x$.

With these two decisions, the mathematical problem of the ambiguity of the $\sqrt{\quad}$ sign disappears, but not the educational problem.

The educational problem

The educational problem has to do with the way in which the decision that mathematicians have taken is transmitted and comes to the students, because students do not learn only what they are told; much of students' learning occurs when they attempt to make sense of the mathematical situations they encounter (Roach, Gibson & Weber. 1994). The

³ For each element in the domain there should be only one element image.

⁴ To be inverted, the function $x \rightarrow x^2$, should fulfill the univalent requirement. So, it has to be confined to a positive or negative range.

⁵ See, for example, Lang, 1971. p. 10

situations that students find in their experience, when learning, are the ones that appear in textbooks they use.

But what do the textbooks say? Spanish textbooks reflect the ambiguity observed in Euler's text. Usually, the lesson about the square root begins with a generalisation of the arithmetic definition, followed by the denomination of new symbols introduced: " $\sqrt[n]{a} = b$ if $b^n = a$; $\sqrt[n]{a}$, is called radical; a , radicand, and n the root's index" (Anaya, 2004, p. 52). Following on from the definition, it is said that the square root and the power of two are inverse operations. So: $\sqrt{b^2} = b$

It is also then said that when the power is even for a positive number there are two numbers that agree with the definition, and therefore $\sqrt{36} = \pm 6$

But, the + sign can be omitted, such that $\sqrt{4}$ "only refers to the positive root: $\sqrt{4}=2$ " (Anaya, 2004, p. 52). However, to distinguish negative roots, the - sign is written before the radical: $-\sqrt{a}$

In addition, in modern textbooks it is said that to solve equations of the type $x^2=b$ there are two ways to focus on this. In the first one, x is directly cleared by finding its value, and in the second one, the square root is taken on both sides of the equals sign.

In both ways there are some textbooks that show traces of the historical and epistemological conceptions previously identified. These are:

1.a) Cleared by finding the value without writing the \pm sign in front of the radical (ignoring Lacroix's rule: $x^2 = b \rightarrow x = \pm\sqrt{b}$) but putting it in front of its numerical value as in Euler's text.

Example: " $2x^2 - 8 = 0$ cleared $x : x^2 = \frac{8}{2} = 4 \rightarrow x = \sqrt{4} = \pm 2$ " (Santillana, 1999, p. 64).

1.b) Finding the value and writing the \pm sign in front of the radical, as in Lacroix's rule.

Example: " $3x^2 - 48 = 0 \rightarrow 3x^2 = 48 \rightarrow x^2 = 16 \rightarrow x = \pm\sqrt{16} = \pm 4$ " (Anaya, 2004, p. 102).

2. Taking the square root on both sides of the equals sign without putting the \pm sign in front of either of the radical signs that are obtained from it, in a duality sense of use, different when it goes with a number or with a letter.

Example: " $(x+3)^2 = 169$. We extract the square root of two members:

$$\sqrt{(x+3)^2} = \sqrt{169} \rightarrow x+3 = \pm 13$$
 (Santillana, 2003, p. 47).

Although the answer is correct in the three options, one cannot say the same of the development shown, since they omit intermediate steps that do not take into account the current mathematical definitions: $\sqrt{4}=2$ and $\sqrt{x^2} = |x|$, which are necessary to avoid conceptual and operative misunderstandings experienced by students, something that often goes unnoticed by teachers and textbooks authors.

For example, the sequence 1.a, $x^2 = 4 \rightarrow x = \sqrt{4} = \pm 2$, reinforces the notion that $\sqrt{x^2} = x$ and $\sqrt{4} = \pm 2$, and does not take into account that $\sqrt{x^2} = |x|$ and $\sqrt{4}=2$. According to Roach, et al. (2004), the omitted steps are as follows:

$$x^2 = 4 \rightarrow \sqrt{x^2} = \sqrt{4} \rightarrow |x| = 2 \rightarrow x = \pm 2$$

The sequence 1.b, $x^2 = 16 \rightarrow x = \pm\sqrt{16} = \pm 4$ also reinforces the notion that $\sqrt{x^2} = x$, and the omitted steps are $x^2 = 16 \rightarrow \sqrt{x^2} = \sqrt{16} \rightarrow |x| = 4 \rightarrow x = \pm 4$

The sequence 2, $(x+3)^2 = 169 \rightarrow \sqrt{(x+3)^2} = \sqrt{169} \rightarrow x+3 = \pm 13$, again reinforces $\sqrt{x^2} = x$ and $\sqrt{169} = \pm 13$. In this case, the omitted steps are:

$$(x+3)^2 = 169 \rightarrow \sqrt{(x+3)^2} = \sqrt{169} \rightarrow |x+3| = 13 \rightarrow x+3 = \pm 13$$

Conclusions

The square root symbol presents educational and mathematical problems with historic origins. Although the problem does not exist from the point of view of current developments in mathematics, one cannot say the same from the point of view of mathematical education.

On the contrary, in our study we have identified in current and representative Spanish textbooks misunderstandings related with historical and epistemological conceptions and sense in the use of the “radical” sign that are deeply rooted.

A review of Spanish texts shows that the teaching proposal reflects the ambiguity of the radical sign used in the expression $\sqrt{4} = \pm 2$, and does not take into account the current mathematical development $\sqrt{x^2} = |x|$ and $\sqrt{4} = 2$.

Finally, the important educational implication that should be pointed out is that in any educational proposal that aims to avoid misunderstandings such as the one evidenced here, the formal definition of radical must be considered, and it must be ensured that students understand the reasons for this definition.

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