Didactical and Epistemological Issues Related to the Concept of Proof

Some mathematics teachers' ideas about the role of proof in Greek secondary curriculum

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Abstract

The paper first examines some epistemological issues concerning the teaching, understanding and production of demonstrative methods. Such issues are the necessity of using proofs, the difference between logical certification and obviousness of geometric figures, as well as the different epistemological meanings of proof, connected either with incomplete argumentations, which however lead to obvious results, or with the logic of non-contradiction to an axiomatic system, which finally persuades.

The paper continues with a study about the role of proof in the Greek secondary curriculum, and investigates the opinions of mathematics teachers about the necessity of teaching demonstration methods. The pressure because of a huge mathematical content, especially in the upper Greek secondary education, leads to the abandonment of many theorem proofs both in Analysis and Geometry. This situation causes a disagreement amongst the mathematics teachers' community over the belief that the main function of proof is the development of rational thinking and the belief that the use of too many and too difficult proofs cause problems in understanding and learning mathematics.

1 The historical role of proof in mathematics

"*An examination of the philosophy and history of mathematics make it clear to me, first of all, that there long have been and still are conflicting opinions on the role of proof in mathematics and in particular on what makes a proof acceptable*"
(Hanna 2000, p. 6)

The existence of different forms and roles of proof through centuries is substantially related to a conflict between two different meanings of proof: "**enlighten**" and "**persuade**"[1]

[1] The term **"enlighten"** refers to demonstrative procedures where, according to a mathematician, the arguments are rather incomplete with logical gaps, but which make the result obvious and finally certain, namely they enlighten through evidence of the figure or of the senses and therefore they finally persuade. According to the contemporary perception of proof, the term **"persuade"** is referred to the logic of non-contradiction within an axiomatic system, which for the mathematician constitutes a testimony (Barbin 1989).

as regards its relation to empiricism and the evidence of senses and to rigor and formalization correspondingly. This conflict became crucial in two different historical periods: (a) The Greek period from the 6th to 4th century B.C. and (b) the period from the 17th to 19th century.

(a) The first part of the Greek period (6^{th}–5^{th} century B.C: logical mathematics) coincides with the born and development of democracy in Greece and is closely related to the development of Greek philosophy. The first demonstrative methods, under the influence of Sophists and Pythagoreans are rather **intuitive and empirical** (Lloyd, 1979) and the older type seems to be **the concrete visual representation** (Szab, 1973). During the second part of the Greek period (4^{th}–3^{rd} century B.C.: deductive and axiomatic mathematics) we have the first epistemological rupture in proof and axiomatic foundations of mathematics. Under the influence of Eleatic philosophy and Plato the need of handling ideal objects raised and the methods became more rigor and anti-empiricist. The indirect proof based on logic dominated against the concrete visual representation (Szab, 1973, Lloyd, 1979, Høyrup 1990).

(b) A similar epistemological rupture took place after many centuries. During the 17^{th} century proof was closely related to **obviousness** and **evidence of senses**, and its meaning was mainly to "enlighten" (Barbin, 1989). A rupture with this conception of proof appeared in the 18^{th} century and was strengthened by Bolzano (1817); it continued during the 19^{th} century, with the domination of algebra and analysis, the emergence of more rigorous methods and the invention of non Euclidean geometries. In Hilbert's axiomatic foundations of geometry (1899) **obviousness and visualization have no meaning.** Proof according to the formalistic view, is a resulting procedure from a non-contradictable axiomatic system, based on formal logic rules, while everything must be proved. Nowadays curriculums are not formalistic anymore, and a lot of discussion is done among researchers about the obviousness (Barbin, 1989).

2 Epistemological and didactical obstacles in teaching and understanding proof

Several researches verified epistemological obstacles in understanding and producing proofs, which are difficult to overcome (Dreyfus & Hadas 1988, Rezende & Nasser 1994, Harel & Sowder 1996, Driscoll 1982).

a) A main epistemological characteristic of the proof is that **the need to solve a problem** really gives meaning to proof, more than the need for rigor mathematics. It is worth to mention here, that the need to solve the problem of irrationality (the existence of irrational numbers) was historically crucial since it probably caused the use of indirect proof in mathematics. It is possible that ancient Greek mathematicians used this theoretical method since the previous empirical ones (anthyphairesis ad infinitum) failed to determine a common measure for the side and diagonal of a quadrangle. However, this need was not clear in the writings of Greek mathematicians, who avoided showing their secret way of doing mathematics. (Arsac 1991, Høyrup 1990, Lloyd 1979, Barbin 1989, Smith 1911). This situation continues until today, since the traditional teaching of geometry presents only the final product of mathematical invention and neglects the conjectures related to inductive thought (Skemp 1971, Freudental 1971, Schoenfeld 1986, Usiskin 1980). According to Freudenthal (1971) "The deductive structure of traditional geometry has never been a convincing didactical success, ... because its deductivity could not be reinvented by the learner but only imposed" (pp. 417–418). The result is that students cannot perceive the necessity of proofs.

b) Another problem concerns the relation between **reason and sensory perception**, especially in geometry, where the change of geometric objects to ideal ones resulted from the philosophical view of Eleatics and Plato as opposed to Sophists, who were based on the evidence of senses; rather, proof has been achieved after overcoming the epistemological obstacle of the **evidence — obviousness of figures**. This obstacle causes difficulties to the students (Lloyd, 1979, Arsac 1991, Barbin, 1989, Høyrup 1990).

c) Another epistemological obstacle is related to different perceptions between teachers and students for the meaning of proof. For the mathematician, proof is mainly intended to "**persuade**" and is related to deductive reasoning. On the contrary, for the student it is intended to "**enlighten**", namely to verify the obviousness and certainty (see also § 1); therefore students need certain examples to be persuaded or sure that they are not mistaken when they observe or create a proof. (Fischbein 1982, in Hanna 2000, Barbin, 1989).

d) The peculiarity of the Greek educational system creates also special **didactical obstacles**. Heuristic-empirical justification and simple proofs are taught in the lower secondary education (high school, students aged 13–15 years), while in the upper secondary education formal proof is taught mainly through geometry (Lyceum, students aged 16–18 years). In Lyceum however, the pressure of a huge mathematical content and the national university entrance examinations, underestimates the teaching of proof by:

  - abandoning many proofs of theorems;
  - the domination of "exercise-ology", namely the solution of as many exercises as possible;
  - private tutorial lessons which direct students only to what is "useful" for the university entrance examinations.

This situation is characterized by a teacher as *"the Waterloo of contemporary Greek Mathematics education"*.

3 The study

The underestimation of geometry and proof influences the community of Greek teachers of mathematics, whose majority had a formalistic education in the 60's, reinforced by the long Greek tradition in geometry and rigor demonstrative procedures.

Purpose: The purpose of our study was to verify different epistemologies among Greek mathematics teachers about proof and possible influences on their teaching practices.

Participants and data collection: 27 upper secondary school mathematics teachers participated during the school year 2006–2007. They were selected on the basis of their willingness to participate in the study. The study is based on a questionnaire of 16 questions.

Data analysis: Teachers' responses were codified and classified according to different themes-subjects. The classification identified different profiles-opinions among teachers as regards their conceptions of proof in the context of secondary school mathematics.

4 Results

Because of space limitations, this paper includes a part of the qualitative elaboration of the answers, which mainly concern the role of proof and the character of demonstrative methods (rigorous or empirical) in school mathematics.

4.1 Results concerning the role of proof in school mathematics

The analysis of the answers revealed the following five roles of proof:

1. **For developing logical thinking skills**: Many responses show that mathematics teaches strongly consider proof to be a highly valuable teaching subject. Some of them think of proof as something adorable; they use enthusiastic and lyric comments, while the historical references to the Greek origin of the concept are remarkable: *"For making conjectures: put targets, think deeply. Conjecture is a holy moment in Mathematics, as compared to applications. These can be done by a computer, were also done by Babylonians and Egyptians. Proof was born in the same time with philosophy in Athens and the Ionian cities. The Greek mathematician in classic Greece was thinking deeply. It also has advantages like elegance and plainness. Euclid's' proof, for example, that "the number of prime numbers is infinite"is a work of art."*

2. **For understanding and learning mathematics**: *"Students hear about the proposition, they understand it, and they know how it was created. Therefore they will remember it better. The proposition by itself is like a cooking recipe. If you don't cook it, you will never remember it."*

3. **To provide confidence**: Some teachers believe that their own epistemology about the role of proof, namely to "persuade" for the truth of a statement, is also the epistemology of their students: *"Students feel confidence about the validity of what they have been taught."* (see § 2.c)

4. **For practical reasons**: to solve exercises or to have success in examinations. Such answers are indicative of the examinational character of the Greek educational system.

5. **Necessary for every day life**: *"It helps creating citizens who deny accepting any 'information' through senses without doubt. Otherwise they would be the ideal victims of any demagogue."* Such "political" comments have a historical correspondence to the ancient Greek democracy.

We tried to compare these results with those of a similar research concerning the last reform recommendations (NCTM, 2000) in the United States to enhance the role of proof in the classroom (Knuth, 2002). From the Table below it is obvious that the first role of developing logical thinking skills is mentioned by both populations; however it is worth to mention the comments on the Greek origins of the demonstrative methods made by Greek teachers. Some characteristic answers also indicate a correspondence between the second roles for both populations. For the rest roles there seem to be no correspondence between the two populations. USA teachers seem to pay attention to the communication developed in the classroom and to the way students think. Such parameters are been neglected by the traditional teaching practice of Greek teachers. Instead, they mention the role of proof in the social community, out of the context of mathematics.

The role of proof in school mathematics	
Greek teachers	***USA Teachers***
1. Developing logical thinking	1. Developing logical thinking
2. Understanding and learning mathematics	2. Explaining why
3. For every day life (social community)	3. Communicating mathematics (classroom community)
4. Providing confidence	4. Displaying thinking
5. For practical reasons (to solve exercises or succeed to examinations)	5. Creating mathematics knowledge

4.2 Results concerning the character of demonstrative methods in school mathematics

The analysis of the responses in some questions verified a disagreement amongst the mathematics teachers' community over the belief that the main function of proof is the development of rational thinking and the belief that the use of too many and too difficult proofs causes problems in understanding and learning mathematics. This disagreement characterizes two different types among teachers:

a) **Type A**: Here belong teachers with **dogmatic** ideas about the usefulness of rigor demonstrative methods, since they serve the development of rational thinking

b) **Type B:** Here belong **less dogmatic** teachers who recognize the negative influence of many and difficult proofs.

Some other questions, gave us the opportunity to make a further separation of Type B, verifying three main profiles A, B1 and B2. However, this categorization was rather difficult for some teachers, since they present characteristics belonging to 2 or 3 different profiles. This point indicates that the Greek educational system creates disagreements not only among different mathematics teachers, but also among what one teacher likes, what he believes that is right to do and what he finally does. We dare to say that the three main profiles correspond to the different types of what was accepted as proof during its historical development: *more, less or no empirical methods.*

a) **Profile A – dogmatic**: These are teachers having opinions about the role and the meaning of proof, which reflect the views of the 19^{th} century's formalism. They face proof as something perfect or given by God and proof teaching as an important duty like religion. For them the main function of proof is the development of rational thinking, and their absolute and consistent ideas affect their teaching practices. They insist on rigorous formulation of proof: "*Proof 'approximately' does not exist*". Proofs based on technology are considered "little toys" or appropriate only for younger students in high or primary school. They don't seem to realize the way students think, and especially their preference to empirical methods. For them students' errors and non-conventional activities are results of either mindlessness or inadequate study, the procedure of repetition being the only way for improvement.

b) **Profile B1 – less dogmatic:** These are teachers who are rather moderate as regards the rigor of formal methods and the acceptance of visual proofs. However the effects of their own classical education, their long experience of formalistic teaching and the educational system, make them finally act in a similar way with teachers of Type A: *"I would accept it* [the visual proof] *but the underlying hypothesis should be mentioned. I would accept this in parallel to the formal proof"*. It is worth mentioning that in this category belong teachers experienced in applications based on Sketchpad.

c) **Profile B2 – more progressive:** Here belongs a minority of rather progressive teachers (e.g. culturally sophisticated, with studies on the didactics of mathematics, or with teaching experiences abroad or in private education). Their opinions about teaching proof reflect the empiricism of the 17^{th} century; however they substantially appreciate the concept of proof as a mathematical object. Their main characteristic is that they recognize students' inability to understand and accept rigorous demonstrative methods, and that they are open to alternative teaching methods, e.g. induction and visual proofs: "*I like to be in fashion. Times change s, we should change too*", "*Only Mathematicians realize the necessity of proof. Most of the students are satisfied by what* *they intuitively perceive*". Unfortunately, they are forced to teach rigorous demonstrative procedures and methodology for solving exercises: "*I am forced to teach in this way; otherwise students will give up*". However, their fresh ideas somehow affect their teaching and their interaction with students.

The above classification was based on the investigation of several themes-subjects mentioned below. In this work we present result concerning only the first two themes:

- The importance of teaching proofs of theorems
- Students' epistemology: empirical thought
- The use of non conventional demonstrative methods (e.g. incomplete justifications, visual proofs or measuring methods)
- Proof as a mathematical concept and a teaching subject
- The purpose of teaching proof
- Teachers' expectations in proof performance

a) ***The importance of teaching proofs of theorems***

Teachers of Type A believe that all theorem proofs should be taught and even the "obvious" ones, for the following reasons:

- To develop rational thinking
- To show the construction and the logic of mathematics
- To show the general validity of a theorem
- As a basis for future proofs
- To understand and remember the theorem better

Some characteristic answers:

- *"The rational thinking is not developed by simple reference to theorems without proof. This leads to mathematical prescriptions for solving properly chosen exercises. Perhaps in the future a teacher would say: 'I give my word of honor that the theorem is true'".*
- *"The phrase 'The proof is obvious' should not exist in school-books. Nothing is obvious when someone comes for the first time in touch with the inevitable nature and rigor of mathematical proof."*

On the contrary, teachers of Type B don't believe that all theorem proofs should be taught. Some reasons and characteristic answers:

- **Limits of time**: *"... (although) It is a crime that proofs in Analysis should not be taught* [according to the official instructions]. *They should be preferred a million times more than this crazy "exercise-ology".*
- **The national examinations**: *"The aim of mathematics in the Lyceum is the success in the national university entrance exams. If a proof is not virtually possible question* [in these exams], *students are NOT interested."*

- *They are difficult or demanding: "Instead of persuading, these lead to learn by heart. Some times, a draft visual verification gives a better result." "Obvious proofs are meaningful for those who are very deeply in mathematics and are not easily satisfied. Such 'heavy' proofs are like 'heavy drugs', a mathematics 'distortion'. They make children run away."*

b) ***Students' epistemology: empirical thought***

We tried to investigate whether these teachers have appreciated the existence of different epistemologies about proof in their classroom, namely: (a) *the epistemology of mathematics taught*, according to which the proposition is true only because it has been proved and (b) *the epistemology of students* based on the need for empirical verifications (Hanna, 2000). The first question of our questionnaire was used for this purpose:

Question 1: A student tries to find the sum of the angles in a triangle by measuring them, just after the theoretical proof has been completed in the classroom. (a) Why do you think the student reacted this way? (b) What should the teacher do?

Teachers of Profile A believe that this reaction is due either to practical reasons (insufficient attention, ignorance of theory), or to student's previous empirical experience. Sometimes the repetition of the proof procedure is suggested as a way of "accuracy":

- *"You cannot so easily eliminate the empirical method".*
- *"The teacher or other students will repeat the proof. I would say: 'Pay attention! Observe how we do it!'"*

On the contrary, teachers of Profile B2, with more progressive ideas, believe that the student has not understood the general character of proof and suggest the teacher to use a contradiction of student's assertion:

- *"He has not realized that after a proposition is proved, it becomes a law."*
- *"The teacher should draw several figures, where measurement would give 179°, 184°... to help student realize that measurement is only an indicative and not a safe method."*

Or they believe that the student does not trust the theoretical proof, or he does not feel the need for proving. The following comments belong to a cultured teacher and "picture" epistemological obstacles:

- *"In secondary education we should not insist on demonstration procedure, but we should be satisfied when students* ***understand*** *mathematics. I think that only mathematicians realize the necessity of proof, while* ***most students are satisfied by what they intuitively perceive****. After all, mathematics history indicates that* ***the necessity of proof was not always obvious.*** *This was really raised after Euclid; until then, they were mostly satisfied being convinced by their senses. Evidently, something relevant happens when students solve analysis exercises based on figures. I would consider such solutions correct, because the child really understands the idea, although these are not analytical proofs... The teacher can use only his authority, but this is not pedagogically correct. Maybe, propositions being intuitively perceived should not be proved.* ***Thus, the necessity of proof will arise through others where intuition is useless."***

Comment: The 66 % of the answers given in Q1a belong to teachers of Profile A. Such responses indicate how difficult it is for teachers to realize that such reactions mostly characterize students' inability to find relations between real world and mathematical objects, and consequently the inability of proof to persuade the student about the proposition's truth.

5 Discussion

Although our results presented here are mainly based on a qualitative analysis of answers, they indicate four factors which influence the beliefs and behavior of Greek mathematics teachers:

1. **The Greek tradition**: The historical references and the enthusiastic comments indicate that the Greek tradition in geometry and in classical demonstrative methods still exists and remains strong in the minds of the majority of mathematics teachers.

2. **Their education**: Most of them aged 40–55 years have been taught mathematics and especially a lot of geometry in a traditional rigorous way.

3. **Their long experience in teaching formal methods in a traditional way**: When we presented some visual proofs supported by technology to a teacher, her reaction changed three times: from a skepticism concerning her abilities, then to enthusiasm and finally to underestimation of visual methods: *"After 30 years of teaching I am not sure if I could work with these methods... I like it very much! ... Rather, in Lyceum I consider them as games; Lyceum is for more serious things. We — all old teachers — have the same ideas."*

4. **The pressure of the Greek educational system** in the upper secondary school, which creates the phenomenon of "exersize-ology", underestimates the teaching of geometry and proof and places to a secondary position the qualitative characteristics of education, related to understanding rather than learning mathematical techniques.

This situation finally causes:

a) Conflicting opinions amongst the mathematics teachers' community about the meaning and the role of proof and the character of demonstrative methods in school mathematics, which more or less reflect either the empirical, or the formalistic character of proof during its historical development.

b) Conflicting opinions amongst a teacher's preferences or beliefs and his/her final teaching practices.

The existence of external factors (the restrictions imposed by the Greek educational system) and also the verified internal factors (the disagreements and the inconsistency amongst mathematics teachers) create an obscure landscape as regards the character and the role of proof and demonstrative methods in school mathematics and finally have a negative influence on teaching and understanding demonstrative procedures in the Greek upper secondary education.

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